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# Commutation relations for the electromagnetic field in the presence of dielectrics and conductors

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## Abstract

Motivated by recent investigations on the Casimir effect, we work out in detail the commutation relations satisfied by the quantized electromagnetic field in the presence of one or two dielectric slabs, with arbitrary dispersive and dissipative properties. In agreement with results derived by previous authors, we explicitly show that at all points in the empty region between the slabs, including their surfaces, the electromagnetic fields always satisfy free-field canonical equal-time commutation relations. This result is a consequence of general analyticity and detailed fall-off properties at large frequencies satisfied by the reflection coefficients of all real materials. It is also shown that this result is not obtained in the case of conductors, if the latter are modelled as perfect mirrors. In such a case, the free-field form of the commutation relations is recovered only at large distances from the mirrors. Failure of perfect-mirror boundary conditions to reproduce the correct form of the commutation relations near the surfaces of the conductors suggests that caution should be used when these idealized boundary conditions are used in investigations of proximity phenomena originating from the quantized electromagnetic field, like the Casimir effect.

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## 1. Introduction

The interaction of radiation with matter has always been a fascinating subject of investigation, and in fact it is at the root of quantum mechanics, with Planck's work on black body radiation. Even though, after the development of quantum electrodynamics (QED) in the middle years of the last century, all fundamental principles involved in this interaction are undoubtedly well understood at the microscopic level, recent experimental advances have prompted much interest in theoretical studies of the quantized electromagnetic (e.m.) field in close proximity to macroscopic bodies. A thorough understanding of this problem is indeed needed for a

correct interpretation of numerous important proximity phenomena of the e.m. origin that include cavity QED [1], the Casimir effect [2], radiative heat transfer [3], quantum friction [4], the Casimir–Polder interaction of Bose–Einstein condensates with a substrate [5], etc. Apart from the intrinsic interest of these phenomena, it has recently been shown that the quantum fluctuations of the e.m. field surrounding macroscopic bodies, which are at the origin of the Casimir effect, could have exciting application in nanotechnology [6].

The common feature of the above e.m. phenomena is that they all involve several macroscopic bodies and possibly one or more microscopic objects (atoms, ions, etc) placed in a vacuum and separated by distances (typical separations range from a few tens of nanometers to several microns) that, while small from a macroscopic point of view, are still large compared to the interatomic distance in condensed bodies. In such circumstances, the microscopic point of view is not of great help, because the long range character of the e.m. field implies that macroscopically large number of atoms are inevitably involved in the interaction. A much more effective approach would be to describe the influence of the macroscopic bodies on the quantized e.m. field in the vacuum just outside their boundaries, in terms of macroscopic features of the bodies like the electric and/or magnetic permittivities. On physical grounds, one expects that such an approach should be feasible, in certain circumstances at least, because the wavelengths of the e.m. fields participating in these phenomena are expected to be of the order of the bodies' separations and are therefore large on the atomic scale. This being the case, the use of macroscopic response functions of the bodies should be legitimate. An inevitable complication that one faces though, when dealing with macroscopic response functions of real bodies, is that they always display dispersion and absorption. As is it well known, the former feature is mathematically reflected in the fact that response functions depend on the frequency  $\omega$  (we shall neglect spatial dispersion, and therefore we shall not consider the possible dependence of the response functions on the wave-vector  $\mathbf{k}$ ), while the presence of dissipation entails that the response functions have a non-vanishing imaginary part. The existence of absorption, in particular, greatly complicates explicit quantization of the macroscopic e.m. field. Unfortunately, such a difficulty cannot be disposed of by simply neglecting dissipation, because dispersive, real-valued response functions inevitably violate causality, and must therefore be rejected.

Fortunately, though, there exists a way out that avoids the above-mentioned difficulties. This is so because a full quantization of the e.m. field is usually not needed, as the quantities of interest are typically statistical averages of quadratic expressions involving the macroscopic e.m. field. For systems that are in thermodynamic equilibrium, such averages can be expressed in terms of (the imaginary part of) suitable macroscopic response functions, as a result of general fluctuation–dissipation theorems derived in the framework of linear-response theory [7]. This general approach was probably pioneered by Rytov [8] in his investigations of e.m. fluctuations in the presence of macroscopic bodies in thermal equilibrium, and it was later used by Lifshitz [9] in his famous theory of dispersion forces between macroscopic condensed bodies. In one form or another, the fluctuation–dissipation theorem is used in all existing approaches to problems involving the quantized e.m. field in the vicinity of or inside macroscopic bodies. In the seventies of the last century, Agarwal used it as the basis of a systematic investigation of QED in the presence of dielectrics and conductors [10]. For a review of the most recent work, we address the reader to [11, 12] (see also references therein). It is important to note that this approach is not restricted to systems in *global* thermodynamic equilibrium, as it is still valid in systems that are only in *local* thermodynamic equilibrium. This feature permits us to include within the scope of the theory other important phenomena, like radiative heat transfer between closely separated bodies (for a recent review see [3]),

and quantum friction [4]. Recently, the theory has also been applied to the investigation of Casimir–Polder [5, 13] and Casimir [14] forces out of thermal equilibrium.

In this paper, we carefully examine the basic quantum-field-theoretical problem of the equal-time commutation relations satisfied by the quantized e.m. field in the presence of dielectrics and/or conductors, in the framework of the general macroscopic theory described above. Our interest in this problem arose from a paper by Milonni [15] on the Casimir effect, in which it was found that near a perfectly reflecting slab, the transverse vector potential and the electric field satisfy a set of equal-time canonical commutation relations of a different form than the one holding for free fields. This result is quite worrisome, in view of the very fundamental character of commutation relations, because it contradicts one’s expectations based on microscopic theory, and therefore it deserves detailed investigation. Addressing this problem is not only interesting as a matter of principle but it is also important for a better understanding of the numerous proximity phenomena arising from quantum fluctuations of the e.m. field described earlier. In many theoretical investigations of these phenomena, one deals with conductors that are frequently modelled as ideal mirrors. A well-known example of this is provided by original Casimir’s derivation [16] of the effect that goes under his name. It is then important to know to what extent conclusions drawn from the ideal-metal model can be trusted. Indeed Casimir physics offers examples where predictions drawn from the ideal-metal model are in contradiction with those derived by more realistic modelling of the plates. One such example is still much debated as we write, and it is the problem of determining the influence of temperature on the magnitude of the Casimir force between two metallic plates in vacuum. It turns out that the ideal-metal model predicts a thermal force that, for sufficiently large separations between the plates, attains a magnitude which is twice the one calculated on the basis of realistic dielectric models of a conductor, displaying a *finite*, though large, dc conductivity (for a review of this puzzle, see for example [17] and references therein).

We point out that recent investigations on the quantization of the macroscopic e.m. field in the presence of dispersive and dissipative materials [11, 12] indicate that canonical equal-time commutators should be ensured as a rule, once the physical requirements of finite reflectivity and absorption losses by the materials are taken into account. In order to further elucidate this important question in the typical setting of Casimir experiments, in this paper we work out a fully explicit analysis of the commutators for the e.m. field, in the presence of dielectric and/or conducting walls with arbitrary dispersion and dissipative features. With respect to previous works [11, 12], more attention is paid here to the detailed fall-off properties of the reflection coefficients of real materials at high frequencies. Our main result, confirming the findings of [11, 12], is that the canonical equal-time commutation relations satisfied by free e.m. fields are *always* valid at all points between two macroscopic dielectric or conducting slabs, including their surfaces, in full agreement with expectations based on the microscopic theory for a system of charged non-relativistic particles interacting with the e.m. field<sup>1</sup>. We also show that canonical commutation relations do not obtain, however, in the case of conductors, if they are modelled as perfect mirrors. In this case we find that near the conductors, the equal-time commutation relations of the vector potential with the electric field have a different form from the free-field case. Only at points that are sufficiently far from the conductors, the free-fields commutators are recovered. Our results generalize those obtained by Milonni, in the one slab setting, and show that the modified form of the commutation relation entailed by perfect-mirror boundary conditions (b.c.) are indeed an artefact of these idealized b.c., not shared by real materials.

<sup>1</sup> Of course in the empty space outside one plate or between two plates, the fields at points  $(\mathbf{r}, t)$  and  $(\mathbf{r}', t')$  do not satisfy *unequal-time* commutation relations of the same form that holds in free space, for time differences  $|t - t'|$  large enough for a light signal to travel from  $\mathbf{r}$  to  $\mathbf{r}'$ , along a path that hits one of the plates [15].

The paper is organized as follows. In section 2 we recall the basic commutation relations satisfied, within the microscopic theory, by the e.m. field in vacuum and in the presence of charged particles. In section 3 we briefly review some general results of linear response theory, as applied to macroscopic quantum electrodynamics, and derive formulae for the expectation values of the field commutators outside a system of macroscopic bodies, in terms of suitable classical Green's functions. In section 4 we estimate Green's functions for a system of one or two dielectric and/or conducting slabs in vacuum, and in section 5 we use them to calculate the commutation relations satisfied by the e.m. field outside the slabs. In section 6 we consider the case of ideal, perfectly reflecting slabs, while section 7 contains our conclusions. Finally, three appendices conclude the paper.

## 2. Commutation relations for e.m. fields: microscopic theory

In this section we briefly recall well-known properties of the commutation relations satisfied by e.m. fields, in the framework of a *microscopic* theory of non-relativistic matter, where ponderable matter is modelled as a collection of non-relativistic charged particles. Here and afterwards, we work in Gaussian e.m. units, and we adopt the Coulomb gauge. As it is well known, the Coulomb gauge is very convenient for studying problems where matter is non-relativistic, and high-energy processes are neglected, for it allows a clear separation of electrostatic and magnetic couplings. In this gauge, quantization is straightforward (see for example the book [18]). We consider first the case of free fields.

### 2.1. Free fields

In empty space, Maxwell equations imply that the electric field is purely transverse:

$$\mathbf{E}_\perp = -\frac{1}{c} \frac{\partial \mathbf{A}_\perp}{\partial t}, \quad (1)$$

where  $\mathbf{A}_\perp$  is the transverse vector potential

$$\nabla \cdot \mathbf{A}_\perp = 0. \quad (2)$$

The fields  $\mathbf{A}_\perp$  and  $\mathbf{E}_\perp$  satisfy the following well-known equal-time canonical commutation relations:

$$[A_{\perp i}(\mathbf{r}, t), A_{\perp j}(\mathbf{r}', t)] = 0, \quad (3)$$

$$[A_{\perp i}(\mathbf{r}, t), E_{\perp j}(\mathbf{r}', t)] = -4\pi i \hbar c \delta_{ij}^\perp(\mathbf{r} - \mathbf{r}'), \quad (4)$$

$$[E_{\perp i}(\mathbf{r}, t), E_{\perp j}(\mathbf{r}', t)] = 0, \quad (5)$$

where  $\delta_{ij}^\perp(\mathbf{x})$  is the transverse delta function<sup>2</sup>:

$$\delta_{ij}^\perp(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) e^{i\mathbf{k} \cdot \mathbf{x}}, \quad (6)$$

with  $k = |\mathbf{k}|$ .

<sup>2</sup> For a review of the properties of the transverse delta function, the reader may consult the book [18].

## 2.2. *e.m. fields coupled to charged particles*

When charged particles are present, the phase space of the total system includes, besides the transverse e.m. fields  $\mathbf{A}_\perp$  and  $\mathbf{E}_\perp$ , the positions  $\mathbf{x}^{(\alpha)}$ , the conjugate momenta  $\mathbf{p}^{(\alpha)}$  and the spins  $\mathbf{s}^{(\alpha)}$  of the particles (labelled by the index  $\alpha$ ). They satisfy the standard (equal-time) commutation relations of non-relativistic quantum mechanics:

$$[x_i^{(\alpha)}, p_j^{(\beta)}] = i\hbar \delta_{ij} \delta_{\alpha\beta}, \quad (7)$$

$$[s_i^{(\alpha)}, s_j^{(\beta)}] = i\hbar \epsilon_{ijk} s_k^{(\alpha)} \delta_{\alpha\beta}, \quad (8)$$

with all other commutators vanishing. In particular  $x_i^{(\alpha)}$ ,  $p_i^{(\alpha)}$  and  $s_i^{(\alpha)}$  all commute with the transverse e.m. fields  $\mathbf{A}_\perp$  and  $\mathbf{E}_\perp$ . Finally  $\mathbf{A}_\perp$  and  $\mathbf{E}_\perp$  satisfy the same equal-time commutation relations holding in empty space, equations (3)–(5).

When charges are present, the electric field  $\mathbf{E}$  also has a longitudinal component  $\mathbf{E}_\parallel$ :

$$\mathbf{E} = \mathbf{E}_\parallel + \mathbf{E}_\perp, \quad (9)$$

where  $\mathbf{E}_\perp$  is still given by equation (1), while  $\mathbf{E}_\parallel$  is equal to

$$\mathbf{E}_\parallel(\mathbf{r}, t) = -\nabla U(\mathbf{r}, t), \quad (10)$$

where  $U$  is the scalar potential:

$$U(\mathbf{r}, t) = \sum_{\alpha} \frac{e^{(\alpha)}}{|\mathbf{r} - \mathbf{x}^{(\alpha)}(t)|}, \quad (11)$$

with  $e^{(\alpha)}$  the charge of particle  $\alpha$ . The scalar potential has to be regarded as a function of the particles positions, and therefore it is not an independent degree of freedom of the system. Since the particle positions commute among themselves and with the transverse fields,  $\mathbf{A}_\perp$  and  $\mathbf{E}_\perp$ , it follows that

$$[U(\mathbf{r}, t), U(\mathbf{r}', t)] = 0, \quad (12)$$

$$[U(\mathbf{r}, t), A_{\perp j}(\mathbf{r}', t)] = 0, \quad (13)$$

$$[U(\mathbf{r}, t), E_{\perp j}(\mathbf{r}', t)] = 0. \quad (14)$$

The above equations imply that the equal-time commutation relations (equations (3)–(5)) remain valid, irrespective of the number and positions of the charged particles, if we replace everywhere the transverse electric field  $E_{\perp j}$  by the total electric field  $E_j$ :

$$[A_{\perp i}(\mathbf{r}, t), A_{\perp j}(\mathbf{r}', t)] = 0, \quad (15)$$

$$[A_{\perp i}(\mathbf{r}, t), E_j(\mathbf{r}', t)] = -4\pi i\hbar c \delta_{ij}^\perp(\mathbf{r} - \mathbf{r}'), \quad (16)$$

$$[E_i(\mathbf{r}, t), E_j(\mathbf{r}', t)] = 0. \quad (17)$$

The obvious conclusion that can be drawn from these elementary remarks is that, *within the microscopic theory*, the canonical equal-time commutation relations satisfied by the e.m. fields (equations (3)–(5) or, alternatively, equations (15)–(17)) should always be valid, and therefore they should hold, in particular, inside a cavity made of an arbitrary material.

### 3. Commutation relations for e.m. fields: macroscopic theory

In this section, we recall a few basic formulae from linear-response theory and we discuss the type of probes that are needed in order to obtain the commutation relations satisfied by the *macroscopic* e.m. field in the presence of dielectrics and conductors. For a review of linear-response theory we address the reader to [7].

In linear-response theory, one considers a quantum-mechanical system, characterized by a (time-independent) Hamiltonian  $H_0$ , in a state of thermal equilibrium described by the density matrix  $\rho$ :

$$\rho = e^{-\beta H} / \text{tr}(e^{-\beta H}), \quad (18)$$

where  $\beta = 1/(k_B T)$ , with  $k_B$  being the Boltzmann constant and  $T$  the temperature. The system is then perturbed by an external perturbation of the form

$$H_{\text{ext}} = - \int d^3 \mathbf{r} \sum_j Q_j(\mathbf{r}, t) f_j(\mathbf{r}, t), \quad (19)$$

where  $f_j(\mathbf{r}, t)$  are the external classical forces, and  $Q_j(\mathbf{r}, t)$  is the dynamical variable of the system conjugate to the force  $f_j(\mathbf{r}, t)$ . One may assume, without loss of generality, that the equilibrium values of the quantities  $Q_j(\mathbf{r}, t)$  all vanish:  $\langle Q_j(\mathbf{r}, t) \rangle = 0$ . The presence of the external forces causes a deviation  $\delta \langle Q_i(\mathbf{r}, t) \rangle$  of the expectation values of  $Q_j(\mathbf{r}, t)$  from their equilibrium values. If the forces  $f_j(\mathbf{r}, t)$  are sufficiently weak,  $\delta \langle Q_i(\mathbf{r}, t) \rangle$  can be taken to be linear functionals of the applied forces  $f_j(\mathbf{r}, t)$  and one may write

$$\delta \langle Q_i(\mathbf{r}, t) \rangle = \sum_j \int d^3 \mathbf{r}' \int_{-\infty}^t dt' \phi_{ij}(\mathbf{r}, \mathbf{r}', t - t') f_j(\mathbf{r}', t'). \quad (20)$$

The above equation assumes that the system was in equilibrium at  $t = -\infty$ , and that it reacts to the external force in a causal way. The quantities  $\phi_{ij}(\mathbf{r}, \mathbf{r}', t - t')$  are called response functions of the system. In principle, they can be measured by applying to the system of interest suitable external classical probes.

By a straightforward computation in time-dependent perturbation theory one may prove that the response functions  $\phi_{ij}(\mathbf{r}, \mathbf{r}', t - t')$  are related to the equilibrium (i.e. in the absence of the external forces) expectation values of the commutators of the dynamical variables  $Q_i(\mathbf{r}, t)$ :

$$\phi_{ij}(\mathbf{r}, \mathbf{r}', t - t') = \frac{i}{\hbar} \langle [Q_i(\mathbf{r}, t), Q_j(\mathbf{r}', t')] \rangle \theta(t - t'), \quad (21)$$

where  $\theta(x)$  is the Heaviside step function ( $\theta(x) = 1$  for  $x > 0$ ,  $\theta(x) = 0$  for  $x < 0$ ) and  $Q_i(\mathbf{r}, t)$  is the Heisenberg operator:

$$Q_i(\mathbf{r}, t) = e^{iH_0 t / \hbar} Q_i(\mathbf{r}, 0) e^{-iH_0 t / \hbar}. \quad (22)$$

As it is well known, equation (21) is the starting point from which several general fluctuation-dissipation theorems can be derived that allow us to express the (symmetrized) correlation functions of the quantities  $Q_i(\mathbf{r}, t)$  in terms of the dissipative component of the response functions  $\phi_{ij}$ . Since we shall not make use of these theorems in what follows, we shall not present them here, and we address the interested reader to [7] for details.

We wish to exploit equation (21) to study the commutation relations satisfied by the *macroscopic* e.m. field at points placed outside a number of dielectric or conducting bodies. Coherently with the spirit of a macroscopic approach, the dielectrics and the conductors will be described in terms of the appropriate electric and magnetic susceptibilities. We suppose from now on that the bodies are made of non-magnetic ( $\mu = 1$ ), isotropic and spatially non-dispersive materials, characterized by a frequency-dependent electric

permittivity  $\epsilon(\mathbf{r}, \omega)$ . We also assume that the bodies have sharp boundaries, and are homogeneous, in such a way that the permittivity  $\epsilon(\mathbf{r}, \omega)$  is independent of  $\mathbf{r}$  within the volume occupied by each body, with discontinuities occurring only at the bodies' interfaces. The e.m. fields satisfy the usual b.c. of macroscopic electrodynamics, namely (i) tangential components of  $\mathbf{E}$  and  $\mathbf{H}$  and (ii) normal components of  $\mathbf{D}$  and  $\mathbf{B}$  must be continuous across the bodies' interfaces. It is opportune at this point to make a remark on the validity of the Coulomb gauge in the presence of dielectrics with sharp boundaries. We note that for non-magnetic materials, which we only consider in this paper, the above b.c. on the e.m. fields imply that the vector potential and its first derivatives are *continuous* across the boundaries of the bodies [19]. As a result, the transversality condition (equation (2)) is meaningful at all points of space, including points on the surfaces of the bodies. Therefore, even in the presence of dielectrics with sharp boundaries, the Coulomb gauge can be fully enforced (this is not true, however, for perfect conductors. See remarks at the end of section 6).

Following Agarwal [10], we now take the external probes to be a system of classical electric and magnetic dipoles, with densities  $\mathbf{P}(\mathbf{r}, t)$  and  $\mathbf{M}(\mathbf{r}, t)$ , respectively, placed outside the bodies. The external Hamiltonian  $H_{\text{ext}}$  is then of the form

$$H_{\text{ext}} = - \int d^3\mathbf{r} [\mathbf{P}^{(\text{ext})}(\mathbf{r}, t) \cdot \mathbf{E}(\mathbf{r}, t) + \mathbf{M}^{(\text{ext})}(\mathbf{r}, t) \cdot \mathbf{B}(\mathbf{r}, t)]. \quad (23)$$

It is convenient for our purposes to have distinct probes for the longitudinal and the transverse components of the e.m. field. This can be achieved by demanding that  $\mathbf{P}^{(\text{ext})}$  be curl free:

$$\nabla \times \mathbf{P}^{(\text{ext})} = \mathbf{0}. \quad (24)$$

If we now express in equation (23) the e.m. field in terms of the scalar and vector potentials

$$\mathbf{E} = -\nabla U - \frac{1}{c} \frac{\partial \mathbf{A}_{\perp}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}_{\perp}, \quad (25)$$

after an integration by parts and exploiting equation (24), the external Hamiltonian can be rewritten as

$$H_{\text{ext}} = \int d^3\mathbf{r} \left[ U(\mathbf{r}, t) \rho^{(\text{ext})}(\mathbf{r}, t) - \frac{1}{c} \mathbf{A}_{\perp}(\mathbf{r}, t) \cdot \mathbf{j}_{\perp}^{(\text{ext})}(\mathbf{r}, t) \right], \quad (26)$$

where  $\rho^{(\text{ext})} = -\nabla \cdot \mathbf{P}^{(\text{ext})}$  and  $\mathbf{j}_{\perp}^{(\text{ext})} = c \nabla \times \mathbf{M}^{(\text{ext})}$ . Note that the current  $\mathbf{j}_{\perp}^{(\text{ext})}$  is transverse:

$$\nabla \cdot \mathbf{j}_{\perp}^{(\text{ext})} = 0. \quad (27)$$

We remark once again that the scalar potential  $U(\mathbf{r}, t)$ , in the external Hamiltonian equation (26) does not represent an independent dynamical variable, and it must be regarded as a function of the particle's position, according to equation (11). Therefore, in the absence of matter, no such term is present in the external Hamiltonian, and the scalar potential is zero.

The response functions are then computed by solving the classical *macroscopic* Maxwell equations with  $\rho^{(\text{ext})}$  and  $\mathbf{j}_{\perp}^{(\text{ext})}$  as external sources:

$$\nabla \cdot \mathbf{D} = 4\pi \rho^{(\text{ext})}, \quad (28)$$

$$\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0, \quad (29)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (30)$$

$$\nabla \times \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c} \mathbf{j}_{\text{tot}}^{(\text{ext})}, \quad (31)$$

where

$$\mathbf{j}_{\text{tot}}^{(\text{ext})} = \mathbf{j}_{\perp}^{(\text{ext})} + \frac{\partial \mathbf{P}^{(\text{ext})}}{\partial t}. \quad (32)$$



The above equations must be solved, subject to the standard b.c. described earlier. By virtue of homogeneity of the bodies, and of linearity of the b.c. at the bodies' interface, the field equations for the scalar potential  $U(\mathbf{r}, t)$  are completely decoupled from those for the transverse vector potential  $\mathbf{A}_\perp(\mathbf{r}, t)$ . Therefore, we have two independent sets of Green's functions:

$$U(\mathbf{r}, t) = \int_{-\infty}^t dt' \int d^3\mathbf{r}' G(\mathbf{r}, \mathbf{r}', t - t') \rho^{(\text{ext})}(\mathbf{r}', t'), \quad (33)$$

$$\mathbf{A}_\perp(\mathbf{r}, t) = \frac{1}{c} \int_{-\infty}^t dt' \int d^3\mathbf{r}' \mathbf{G}_\perp(\mathbf{r}, \mathbf{r}', t - t') \cdot \mathbf{j}_\perp^{(\text{ext})}(\mathbf{r}', t'), \quad (34)$$

where  $\mathbf{G}_\perp(\mathbf{r}, \mathbf{r}', t - t')$  has to be understood as a dyadic Green's function.

From the general result of linear-response theory, equation (21), we then obtain the following expressions for the two-times expectation values of the commutators of the e.m. potentials:

$$\langle [U(\mathbf{r}, t), U(\mathbf{r}', t')] \rangle = i\hbar G(\mathbf{r}, \mathbf{r}', t - t'), \quad (35)$$

$$\langle [U(\mathbf{r}, t), A_{\perp i}(\mathbf{r}', t')] \rangle = 0, \quad (36)$$

$$\langle [A_{\perp i}(\mathbf{r}, t), A_{\perp j}(\mathbf{r}', t')] \rangle = -i\hbar G_{\perp ij}(\mathbf{r}, \mathbf{r}', t - t'), \quad (37)$$

where  $t > t'$ . For our purposes, it is convenient to split Green's functions, *outside* the bodies, as sums of an empty-space contribution plus a correction arising from the material bodies:

$$G(\mathbf{r}, \mathbf{r}', t - t') = G^{(0)}(\mathbf{r} - \mathbf{r}', t - t') + F^{(\text{mat})}(\mathbf{r}, \mathbf{r}', t - t'), \quad (38)$$

and

$$\mathbf{G}_\perp(\mathbf{r}, \mathbf{r}', t - t') = \mathbf{G}_\perp^{(0)}(\mathbf{r} - \mathbf{r}', t - t') + \mathbf{F}_\perp^{(\text{mat})}(\mathbf{r}, \mathbf{r}', t - t'). \quad (39)$$

Here,  $G^{(0)}$  and  $\mathbf{G}_\perp^{(0)}$  denote Green's functions in free space, while  $F^{(\text{mat})}$  and  $\mathbf{F}_\perp^{(\text{mat})}$  describe the effects resulting from the presence of the bodies. Such a splitting presents the advantage that all singularities are included in the free parts  $G^{(0)}$  and  $\mathbf{G}_\perp^{(0)}$ , while the quantities  $F^{(\text{mat})}$  and  $\mathbf{F}_\perp^{(\text{mat})}$  are smooth ordinary functions of  $\mathbf{r}$  and  $\mathbf{r}'$ . The free-field Green's functions have the following well-known expressions:

$$G^{(0)}(\mathbf{r} - \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta(t - t') \quad (40)$$

and

$$G_{\perp ij}^{(0)} = c \int \frac{d^3\mathbf{k}}{2\pi^2 k} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) e^{i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')} \sin[kc(t - t')]. \quad (41)$$

The factor  $\delta(t - t')$  in the expression of  $G^{(0)}$  expresses the instantaneous character of the longitudinal electric field in the Coulomb gauge. The expressions for the equal-time commutators of the e.m. fields are easily derived by taking suitable limits of equations (35)–(37) and of their time derivatives, for  $t \rightarrow t'^+$ . Upon using equations (38) and (39), and exploiting the following three relations that are obvious consequences of equation (1):

$$\langle [U(\mathbf{r}, t), E_{\perp i}(\mathbf{r}', t')] \rangle = -\frac{1}{c} \frac{\partial}{\partial t'} \langle [U(\mathbf{r}, t), A_{\perp i}(\mathbf{r}', t')] \rangle, \quad (42)$$

$$\langle [A_{\perp i}(\mathbf{r}, t), E_{\perp j}(\mathbf{r}', t')] \rangle = -\frac{1}{c} \frac{\partial}{\partial t'} \langle [A_{\perp i}(\mathbf{r}, t), A_{\perp j}(\mathbf{r}', t')] \rangle \quad (43)$$

and

$$\langle [E_{\perp i}(\mathbf{r}, t), E_{\perp j}(\mathbf{r}', t')] \rangle = \frac{1}{c^2} \frac{\partial^2}{\partial t \partial t'} \langle [A_{\perp i}(\mathbf{r}, t), A_{\perp j}(\mathbf{r}', t')] \rangle, \quad (44)$$

from equations (35)–(37) we obtain

$$\langle [U(\mathbf{r}, t), U(\mathbf{r}', t)] \rangle = i\hbar A^{(\text{mat})}(\mathbf{r}, \mathbf{r}'), \quad (45)$$

$$\langle [U(\mathbf{r}, t), A_{\perp i}(\mathbf{r}', t)] \rangle = 0, \quad (46)$$

$$\langle [U(\mathbf{r}, t), E_{\perp i}(\mathbf{r}', t)] \rangle = 0, \quad (47)$$

$$\langle [A_{\perp i}(\mathbf{r}, t), A_{\perp j}(\mathbf{r}', t)] \rangle = -i\hbar A_{\perp ij}^{(\text{mat})}(\mathbf{r}, \mathbf{r}'), \quad (48)$$

$$\langle [A_{\perp i}(\mathbf{r}, t), E_{\perp j}(\mathbf{r}', t)] \rangle = -4\pi i\hbar c \delta_{ij}^{\perp}(\mathbf{r} - \mathbf{r}') + \frac{i\hbar}{c} B_{\perp ij}^{(\text{mat})}(\mathbf{r}, \mathbf{r}'), \quad (49)$$

and

$$\langle [E_{\perp i}(\mathbf{r}, t), E_{\perp j}(\mathbf{r}', t)] \rangle = -\frac{i\hbar}{c^2} C_{\perp ij}^{(\text{mat})}(\mathbf{r}, \mathbf{r}'), \quad (50)$$

where we defined

$$A^{(\text{mat})}(\mathbf{r}, \mathbf{r}') \equiv \lim_{t \rightarrow t'^+} F^{(\text{mat})}(\mathbf{r}, \mathbf{r}', t - t'), \quad (51)$$

$$A_{\perp ij}^{(\text{mat})}(\mathbf{r}, \mathbf{r}') \equiv \lim_{t \rightarrow t'^+} F_{\perp ij}^{(\text{mat})}(\mathbf{r}, \mathbf{r}', t - t'), \quad (52)$$

$$B_{\perp ij}^{(\text{mat})}(\mathbf{r}, \mathbf{r}') \equiv \lim_{t \rightarrow t'^+} \frac{\partial F_{\perp ij}^{(\text{mat})}}{\partial t'}(\mathbf{r}, \mathbf{r}', t - t') \quad (53)$$

and

$$C_{\perp ij}^{(\text{mat})}(\mathbf{r}, \mathbf{r}') \equiv \lim_{t \rightarrow t'^+} \frac{\partial^2 F_{\perp ij}^{(\text{mat})}}{\partial t \partial t'}(\mathbf{r}, \mathbf{r}', t - t'). \quad (54)$$

By comparing equations (45)–(50) with equations (3)–(5) and equations (12)–(14), we see that outside the bodies the free-field canonical commutation relations are recovered provided that the quantities  $A^{(\text{mat})}$ ,  $A_{\perp ij}^{(\text{mat})}$ ,  $B_{\perp ij}^{(\text{mat})}$  and  $C_{\perp ij}^{(\text{mat})}$  are zero. We shall prove below that this is indeed the case, as a result of analyticity and fall-off properties at large frequencies of the reflection coefficients of all real materials.

Before we turn to detailed computations, we present below the field equations satisfied by  $G$  and  $\mathbf{G}_{\perp}$ . They are conveniently expressed in terms of the (one-sided) Fourier transforms of Green's functions, defined as

$$\tilde{G}(\mathbf{r}, \mathbf{r}', \omega) = \int_0^{\infty} dt G(\mathbf{r}, \mathbf{r}', t) e^{i\omega t}, \quad (55)$$

$$\tilde{\mathbf{G}}_{\perp}(\mathbf{r}, \mathbf{r}', \omega) = \int_0^{\infty} dt \mathbf{G}_{\perp}(\mathbf{r}, \mathbf{r}', t) e^{i\omega t}. \quad (56)$$

From Maxwell equations we then obtain

$$\nabla \cdot [\epsilon(\mathbf{r}, \omega) \nabla \tilde{G}] = -4\pi \delta(\mathbf{r} - \mathbf{r}'), \quad (57)$$

$$(\Delta + \epsilon(\mathbf{r}, \omega) \omega^2 / c^2) \tilde{\mathbf{G}}_{\perp}(\mathbf{r}, \mathbf{r}', \omega) = -4\pi \delta_{\perp}(\mathbf{r} - \mathbf{r}'), \quad (58)$$

where  $\delta_{\perp}(\mathbf{r} - \mathbf{r}')$  is the transverse delta-function dyad, equation (6). These equations must be solved with the appropriate b.c. at the bodies' interfaces, and must be subject to the conditions required for a retarded Green's function [20]. For later use, it is useful to recall the main properties enjoyed by Green's functions [21]. First of all, they satisfy the following *reciprocity* relations:

$$\tilde{G}(\mathbf{r}, \mathbf{r}', \omega) = \tilde{G}(\mathbf{r}', \mathbf{r}, \omega) \quad (59)$$

and

$$\tilde{G}_{\perp ij}(\mathbf{r}, \mathbf{r}', \omega) = \tilde{G}_{\perp ji}(\mathbf{r}', \mathbf{r}, \omega), \quad (60)$$

which are a consequence of microscopic reversibility. The next set of properties express reality features of Green's functions, and are a direct consequence of reality of the external sources

$$\tilde{G}^*(\mathbf{r}, \mathbf{r}', \omega) = \tilde{G}(\mathbf{r}, \mathbf{r}', -\omega), \quad (61)$$

and

$$\tilde{G}_{\perp ij}^*(\mathbf{r}, \mathbf{r}', \omega) = \tilde{G}_{\perp ij}(\mathbf{r}, \mathbf{r}', -\omega). \quad (62)$$

The next set of properties is a consequence of the fact that the permittivity  $\epsilon(\omega)$  of any causal medium is an analytic function of the frequency  $w$  in the upper complex half-plane  $\mathcal{C}^+$  [22] (see also appendix B). This implies that Green's functions  $\tilde{G}(\mathbf{r}, \mathbf{r}', \omega)$  and  $\tilde{\mathbf{G}}_{\perp}(\mathbf{r}, \mathbf{r}', \omega)$  are also *analytic* in  $\mathcal{C}^+$ , as it must be the case for a retarded response function. In  $\mathcal{C}^+$  they satisfy the conditions  $\tilde{G}^*(\mathbf{r}, \mathbf{r}', w) = \tilde{G}(\mathbf{r}, \mathbf{r}', -w^*)$  and  $\tilde{G}_{\perp ij}^*(\mathbf{r}, \mathbf{r}', w) = \tilde{G}_{\perp ij}(\mathbf{r}, \mathbf{r}', -w^*)$  that generalize the reality conditions (equations (61) and (62)), respectively. These more general properties imply that Green's functions are real along the imaginary frequency axis:

$$\tilde{G}(\mathbf{r}, \mathbf{r}', i\xi) = \tilde{G}^*(\mathbf{r}, \mathbf{r}', i\xi), \quad (63)$$

and

$$\tilde{G}_{\perp ij}(\mathbf{r}, \mathbf{r}', i\xi) = \tilde{G}_{\perp ij}^*(\mathbf{r}, \mathbf{r}', i\xi). \quad (64)$$

It is finally useful to write down the inversion formulae expressing Green's functions, in the time domain, in terms of their Fourier transforms. They are

$$G(\mathbf{r}, \mathbf{r}', t - t') = \int_{\Gamma} dw \tilde{G}(\mathbf{r}, \mathbf{r}', w) e^{-iw(t-t')} \quad (65)$$

$$\mathbf{G}_{\perp}(\mathbf{r}, \mathbf{r}', t - t') = \int_{\Gamma} dw \tilde{\mathbf{G}}_{\perp}(\mathbf{r}, \mathbf{r}', w) e^{-iw(t-t')}, \quad (66)$$

where  $\Gamma$  is any contour in  $\mathcal{C}^+$  that can be obtained by smoothly deforming the real frequency axis, keeping fixed the end-points at infinity. Analyticity of Green's functions in  $\mathcal{C}^+$  ensures that the integrals on the rhs are independent of the chosen contour  $\Gamma$ .

In the next two sections, we shall compute the Green's functions at points outside a single dielectric slab, and between two plane parallel slabs.

#### 4. Green's functions outside dielectrics and conductors

In this section, we evaluate the e.m. Green's functions for the scalar and for the transverse vector potentials outside dielectric and/or conducting slabs. We note that Green's functions for the total electric field  $\mathbf{E}$  in a multilayer system have been derived already in [23].

In the next two subsections we shall separately consider the cases of one slab in vacuum, and two plane-parallel slabs separated by an empty gap. We choose our Cartesian coordinate system such that the  $z$ -axis is perpendicular to the slabs. Translational invariance of the system in the  $(x, y)$  plane implies that the quantities  $\tilde{F}^{(\text{mat})}$  and  $\tilde{\mathbf{F}}_{\perp}^{(\text{mat})}$  are functions only of  $z, z'$  and  $(\mathbf{r}_{\perp} - \mathbf{r}'_{\perp})$ , where we denote by  $\mathbf{x}_{\perp}$  the projection of the vector  $\mathbf{x}$  onto the  $(x, y)$  plane. The computation is facilitated if we express  $\tilde{G}^{(0)}$  and  $\tilde{\mathbf{G}}_{\perp}^{(0)}$  in a form that is adapted to the symmetries of our problem. Consider first the free scalar Green's function  $\tilde{G}^{(0)}$ :

$$\tilde{G}^{(0)}(\mathbf{r} - \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|}. \quad (67)$$

We note that  $\tilde{G}^{(0)}$  is independent of the complex frequency  $w$ , as it must be because of the instantaneous character of the longitudinal electric field in the Coulomb gauge. For our purposes, the convenient form of  $\tilde{G}^{(0)}$  is the following well-known Weyl representation:

$$\tilde{G}^{(0)} = \int \frac{d^2\mathbf{k}_{\perp}}{2\pi k_{\perp}} e^{i\mathbf{k}_{\perp} \cdot (\mathbf{r}_{\perp} - \mathbf{r}'_{\perp}) - k_{\perp}|z - z'|}, \quad (68)$$

that can be easily obtained by integrating over  $k_3$  the standard plane-wave decomposition of  $\tilde{G}^{(0)}$ . The above expression for  $\tilde{G}^{(0)}$  can also be written as

$$\tilde{G}^{(0)} = \int \frac{d^2\mathbf{k}_{\perp}}{2\pi k_{\perp}} e^{i\bar{\mathbf{k}}^{(\pm)} \cdot (\mathbf{r} - \mathbf{r}')}, \quad (69)$$

where we define  $\bar{\mathbf{k}}^{(\pm)} = \mathbf{k}_{\perp} \pm ik_{\perp} \hat{\mathbf{z}}$  and the upper (lower) sign is for  $z \geq z'$  ( $z \leq z'$ ). Consider now the familiar representation of  $\tilde{\mathbf{G}}_{\perp}^{(0)}$ :

$$\tilde{G}_{\perp ij}^{(0)} = \int \frac{d^3\mathbf{k}}{2\pi^2} \frac{1}{k^2 - k_0^2} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) e^{i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')}, \quad (70)$$

where  $k_0 = w/c$ . In appendix A, we show that  $\tilde{\mathbf{G}}_{\perp}^{(0)}$  can be decomposed as the sum of two dyads:

$$\tilde{\mathbf{G}}_{\perp}^{(0)} = \tilde{\mathbf{U}}^{(0)} + \tilde{\mathbf{V}}^{(0)}. \quad (71)$$

Here,  $\tilde{\mathbf{U}}^{(0)}$  denotes the tensor of components

$$\tilde{U}_{ij}^{(0)} = i \int \frac{d^2\mathbf{k}_{\perp}}{2\pi k_z} \left( e_{\perp i} e_{\perp j} + \frac{\xi_i^{(\pm)} \xi_j^{(\pm)}}{k_0^2} \right) e^{i\bar{\mathbf{k}}^{(\pm)} \cdot (\mathbf{r} - \mathbf{r}')}, \quad (72)$$

where  $k_z = \sqrt{k_0^2 - k_{\perp}^2}$  (the square root is defined such that  $\text{Im}(k_z) > 0$ ),  $\mathbf{e}_{\perp} = \hat{\mathbf{z}} \times \hat{\mathbf{k}}_{\perp}$ ,  $\mathbf{k}^{(\pm)} = \mathbf{k}_{\perp} \pm k_z \hat{\mathbf{z}}$  and  $\xi^{\pm} = k_{\perp} \hat{\mathbf{z}} \mp k_z \hat{\mathbf{k}}_{\perp}$ . As to  $\tilde{V}_{ij}^{(0)}$ , it can be written as

$$\tilde{V}_{ij}^{(0)} = \frac{1}{k_0^2} \frac{\partial^2 \tilde{\Psi}^{(0)}}{\partial x_i \partial x'_j}, \quad (73)$$

where  $\tilde{\Psi}^{(0)}$  is the function

$$\tilde{\Psi}^{(0)} = \int \frac{d^2\mathbf{k}_{\perp}}{2\pi k_{\perp}} e^{i\bar{\mathbf{k}}^{(\pm)} \cdot (\mathbf{r} - \mathbf{r}')}. \quad (74)$$

In both equations (72) and (74), the upper (lower) sign is for  $z \geq z'$  ( $z \leq z'$ ). It is useful to provide a simple intuitive interpretation for the above Green's functions that will be useful later when we consider the influence of a material slab. Consider first the expression for  $\tilde{G}^{(0)}$  given in equation (69): we can interpret this as consisting of a superposition of instantaneous scalar waves originating from point  $\mathbf{r}'$  that propagate to the right (left) with the wave-vector  $\bar{\mathbf{k}}^{(+)}$  ( $\bar{\mathbf{k}}^{(-)}$ ). Consider now our expression for  $\tilde{\mathbf{G}}_{\perp}^{(0)}$ , equation (71). Its first contribution  $\tilde{\mathbf{U}}^{(0)}$ ,

equation (72), can be physically interpreted as a superposition of e.m. waves with TE and TM polarization corresponding, respectively, to the first and second term between the round brackets in equation (72). These waves originate from point  $\mathbf{r}'$  and propagate to the right (left) with the wave-vector  $\mathbf{k}^{(+)}$  ( $\mathbf{k}^{(-)}$ ). We note that for  $k_0 > k_\perp$  these modes represent propagating waves, while for  $k_0 < k_\perp$  they are evanescent waves that decay exponentially as we move away from  $z'$ . The second contribution to  $\tilde{\mathbf{G}}_\perp^{(0)}$ ,  $\tilde{\mathbf{V}}^{(0)}$  can instead be interpreted as representing scalar waves that propagate instantaneously from point  $\mathbf{r}'$  in the right (left) direction, with the wave-vector  $\bar{\mathbf{k}}^{(+)}$  ( $\bar{\mathbf{k}}^{(-)}$ ).

We are now ready to compute  $\tilde{F}^{(\text{bodies})}$  and  $\tilde{\mathbf{F}}_\perp^{(\text{bodies})}$ . We consider first the one-slab case.

#### 4.1. The case of one slab

In this section, we compute Green's functions outside a single dielectric or conducting slab, occupying the half-space  $z < 0$ . Following the remarks of the previous section, outside the slab and on its surface, i.e. for  $z, z' \geq 0$ , we define

$$\tilde{G}^{(\text{wall})}(\mathbf{r}, \mathbf{r}', w) = \tilde{G}^{(0)}(\mathbf{r} - \mathbf{r}') + \tilde{F}^{(\text{wall})}(\mathbf{r}_\perp - \mathbf{r}'_\perp, z, z', w) \quad (75)$$

and

$$\tilde{\mathbf{G}}_\perp^{(\text{wall})}(\mathbf{r}, \mathbf{r}', w) = \tilde{\mathbf{G}}_\perp^{(0)}(\mathbf{r} - \mathbf{r}', w) + \tilde{\mathbf{F}}_\perp^{(\text{wall})}(\mathbf{r}_\perp - \mathbf{r}'_\perp, z, z', w). \quad (76)$$

Fixing once and for all  $z' \geq 0$ , we make for  $\tilde{F}^{(\text{wall})}$  the following ansatz:

$$\tilde{F}^{(\text{wall})} = - \int \frac{d^2 \mathbf{k}_\perp}{2\pi k_\perp} \bar{r}(w) e^{i\mathbf{k}_\perp \cdot (\mathbf{r}_\perp - \mathbf{r}'_\perp) - k_\perp(z+z')}, \quad z \geq 0. \quad (77)$$

For  $z < 0$ , the complete Green's function is taken to be of the form

$$\tilde{G}^{(\text{wall})} = \int \frac{d^2 \mathbf{k}_\perp}{2\pi k_\perp} \bar{t}(w) e^{i\mathbf{k}_\perp \cdot (\mathbf{r}_\perp - \mathbf{r}'_\perp) - k_\perp(z'-z)}, \quad z < 0. \quad (78)$$

Both ansatz ensure appropriate fall off for  $|z| \rightarrow \infty$ . It is easy to verify that the above ansatz satisfy the b.c. at  $z = 0$ , provided that we take

$$\bar{r}(w) = \frac{\epsilon(w) - 1}{\epsilon(w) + 1} \quad (79)$$

and

$$\bar{t}(w) = 1 - \bar{r}(w). \quad (80)$$

The chosen forms of  $\tilde{F}^{(\text{wall})}$ , for  $z > 0$ , and  $\tilde{G}^{(\text{wall})}$ , for  $z < 0$ , have a simple physical interpretation that will be useful later when we shall consider the more elaborate case of two slabs. In the empty space, the source  $\tilde{\rho}(\mathbf{r}', w)$  generates 'instantaneous' scalar waves of (complex) frequency  $w$  originating at  $\mathbf{r}'$  and propagating in the right direction (i.e. towards larger  $z$ ) with the (complex) wave-vector  $\bar{\mathbf{k}}^{(+)}$ , and in the left direction with the wave-vector  $\bar{\mathbf{k}}^{(-)}$ . When a wall is present, the left-moving waves hit the wall and then we have a reflected wave with amplitude  $\bar{r}(w)$  and a transmitted wave of amplitude  $\bar{t}(w)$ .

We can now evaluate  $\tilde{\mathbf{F}}_\perp^{(\text{wall})}$ . In a way analogous to equation (71), we decompose it as

$$\tilde{\mathbf{F}}_\perp^{(\text{wall})} = \tilde{\mathbf{U}}^{(\text{wall})} + \tilde{\mathbf{V}}^{(\text{wall})}. \quad (81)$$

Inside the slab, for the full Green's function, we set instead

$$\tilde{\mathbf{G}}_\perp^{(\text{wall})} = \tilde{\mathbf{U}}^{(\text{in})} + \tilde{\mathbf{V}}^{(\text{in})}, \quad z < 0. \quad (82)$$

Linearity of the boundary-value problem permits us to determine separately  $\tilde{\mathbf{U}}^{(\text{wall})}$  and  $\tilde{\mathbf{V}}^{(\text{wall})}$ . The physical picture of  $\tilde{\mathbf{U}}^{(0)}$  as a superposition of TE and TM waves suggests at once the following ansatz for  $\tilde{\mathbf{U}}^{(\text{wall})}$ :

$$\tilde{U}_{ij}^{(\text{wall})} = i \int \frac{d^2 \mathbf{k}_\perp}{2\pi k_z} \left( e_{\perp i} e_{\perp j} r^{(s)}(w, k_\perp) + \frac{\xi_i^{(+)} \xi_j^{(-)}}{k_0^2} r^{(p)}(w, k_\perp) \right) e^{i(\mathbf{k}^{(+)} \cdot \mathbf{r} - \mathbf{k}^{(-)} \cdot \mathbf{r}')}, \quad (83)$$

where  $r^{(s)}(w, \mathbf{k}_\perp)$  and  $r^{(p)}(w, \mathbf{k}_\perp)$  are the familiar Fresnel reflections coefficients for TE and TM waves, respectively:

$$r^{(s)}(w, k_\perp) = \frac{k_z - q}{k_z + q}, \quad (84)$$

$$r^{(p)}(w, k_\perp) = \frac{\epsilon(w)k_z - q}{\epsilon(w)k_z + q}, \quad (85)$$

where  $q = \sqrt{\epsilon(w)k_0^2 - k_\perp^2}$ . A somewhat lengthy solution of the boundary-value problem indeed confirms the above intuitive form of  $\tilde{U}_{ij}^{(\text{wall})}$ . Consider now  $\tilde{\mathbf{V}}^{(\text{wall})}$ . Equation (74) suggests that we set

$$\tilde{V}_{ij}^{(\text{wall})} = \frac{1}{k_0^2} \frac{\partial^2 \Psi^{(\text{wall})}}{\partial x_i \partial x_j'}, \quad (86)$$

while inside the slab (i.e. for  $z < 0$ ) we set

$$\tilde{V}_{ij}^{(\text{in})} = \frac{1}{k_0^2} \frac{\partial^2 \Psi^{(\text{in})}}{\partial x_i \partial x_j'}. \quad (87)$$

It can be seen that the appropriate boundary dielectric conditions at  $z = 0$  are satisfied, provided that the functions  $\tilde{\Psi}^{(0)}$ ,  $\tilde{\Psi}^{(\text{wall})}$  and  $\tilde{\Psi}^{(\text{in})}$  fulfil there the following b.c.:

$$\tilde{\Psi}^{(\text{in})}|_{z=0} = (\tilde{\Psi}^{(0)} + \tilde{\Psi}^{(\text{wall})})|_{z=0}, \quad (88)$$

$$\epsilon(w)\tilde{\Psi}^{(\text{in})'}|_{z=0} = (\tilde{\Psi}^{(0)'} + \tilde{\Psi}^{(\text{wall})'})|_{z=0}, \quad (89)$$

where a prime denotes a derivative with respect to  $z$ . One then finds

$$\tilde{\Psi}^{(\text{wall})} = - \int \frac{d^2 \mathbf{k}_\perp}{2\pi k_\perp} \bar{r}(w) e^{i\mathbf{k}^{(+)} \cdot \mathbf{r} - i\mathbf{k}^{(-)} \cdot \mathbf{r}'}, \quad (90)$$

where  $\bar{r}(w)$  is the reflection coefficient in equation (79). We note that the expression of  $\tilde{\Psi}^{(\text{wall})}$  coincides with that of  $\tilde{F}^{(\text{wall})}$ . We remark that  $\tilde{F}^{(\text{wall})}$  and  $\tilde{\mathbf{F}}_\perp^{(\text{wall})}$  are analytic functions of the frequency  $w$  in the upper complex plane  $\mathcal{C}^+$ , as a result of analyticity in  $\mathcal{C}^+$  of the reflection coefficients  $\bar{r}(w)$ ,  $r^{(\alpha)}(w)$  (see appendix B). Moreover, we note that  $\tilde{F}^{(\text{wall})}$  has no singularities along the real-frequency axis, as it can be easily checked from equation (77), if one considers that the reflection coefficient  $\bar{r}(w)$  is finite in  $\mathcal{C}^+$  (see appendix B). As to  $\tilde{\mathbf{F}}_\perp^{(\text{wall})}$ , it only has an integrable singularity at  $k_z = 0$ . The presence of singular factors proportional to  $k_0^{-2}$  in the expressions of  $\tilde{U}_{ij}^{(\text{wall})}$  and  $\tilde{V}_{ij}^{(\text{wall})}$  (see equations (83) and (86)) does not cause any further singularities at  $w = 0$  for it can be verified that these singular terms cancel each other upon taking the sum of  $\tilde{U}_{ij}^{(\text{wall})}$  and  $\tilde{V}_{ij}^{(\text{wall})}$  as we now show. Indeed, upon collecting in equation (83) and equation (86) the terms that are singular at  $w = 0$ , we obtain

$$\lim_{w \rightarrow 0} \tilde{F}_{\perp ij}^{(\text{wall})}(\mathbf{r}, \mathbf{r}') = c^2 \lim_{w \rightarrow 0} \int \frac{d^2 \mathbf{k}_\perp}{2\pi k_\perp} \frac{r^{(p)} - \bar{r}}{w^2} \bar{k}_i^{(+)} \bar{k}_j^{(-)} e^{i\mathbf{k}^{(+)} \cdot \mathbf{r} - i\mathbf{k}^{(-)} \cdot \mathbf{r}'}, \quad (91)$$

where we made use of the following relations:

$$\boldsymbol{\xi}^{(\pm)} = \mp i \bar{\mathbf{k}}^{(\pm)} + O(w^2), \quad (92)$$

$$\mathbf{k}^{(\pm)} = \bar{\mathbf{k}}^{(\pm)} + O(w^2), \quad (93)$$

$$k_z = i k_{\perp} + O(w^2), \quad (94)$$

to substitute everywhere  $\boldsymbol{\xi}^{(\pm)}$ ,  $\mathbf{k}^{(\pm)}$  and  $k_z$  by  $\mp i \bar{\mathbf{k}}^{(\pm)}$ ,  $\bar{\mathbf{k}}^{(\pm)}$  and  $k_{\perp}$ , respectively. Now in appendix B it is shown that for both dielectrics and conductors, the difference  $r^{(p)} - \bar{r}$  approaches zero as  $w^2$ :

$$r^{(p)}(w) - \bar{r}(w) = O(w^2). \quad (95)$$

Therefore, the ratio  $(r^{(p)} - \bar{r})/w^2$  is finite as  $\omega$  tends to zero, showing that  $\tilde{F}_{\perp ij}^{(\text{wall})}$  is regular at  $w = 0$ .

#### 4.2. The case of two plane-parallel slabs

In this section we calculate Green's functions for the case of a cavity constituted by two non-magnetic homogeneous, isotropic and spatially non-dispersive plane-parallel slabs separated by vacuum. We assume that the slabs can be characterized by the respective electric permittivities,  $\epsilon_1(w)$  and  $\epsilon_2(w)$ . We choose our Cartesian coordinate system in such a way that slab 1 occupies the region  $-\infty < z \leq 0$ , while slab 2 occupies the region  $d \leq z < \infty$ ,  $d$  being the separation between the two slabs. The formulae derived in the preceding section, for the one-slab case, can be easily generalized to the two-slab setting, on the basis of the intuitive physical picture of the free Green's functions as consisting of left- and right-moving waves originating from  $\mathbf{r}'$ .

Let us consider first the scalar Green's function  $\tilde{G}$ . Analogously to what we did in the previous section, inside the cavity (i.e. for  $0 \leq z, z' \leq d$ ) we set

$$\tilde{G}^{(\text{cav})}(\mathbf{r}, \mathbf{r}', w) = \tilde{G}^{(0)}(\mathbf{r} - \mathbf{r}') + \tilde{F}^{(\text{cav})}(\mathbf{r}_{\perp} - \mathbf{r}'_{\perp}, z, z', w). \quad (96)$$

The expression that one finds for  $\tilde{F}^{(\text{cav})}$  is analogous to  $\tilde{F}^{(\text{wall})}$ , but of course one must take account now of the possibility of multiple reflections off the two slabs. This is easily done, by inserting for each reflection by slab  $i$  the appropriate reflection coefficients  $\bar{r}_i(w)$  that has an expression analogous to equation (79) (with  $\epsilon_i(w)$  in the place of  $\epsilon(w)$ ). Moreover, a factor  $e^{-2k_{\perp}d}$  must be included for each round-way trip from one slab to the other and back. One obtains

$$\begin{aligned} \tilde{F}^{(\text{cav})} = \int \frac{d^2 \mathbf{k}_{\perp}}{2\pi k_{\perp}} \left[ \left( \frac{1}{\mathcal{A}} - 1 \right) (e^{i\bar{\mathbf{k}}^{(+)} \cdot (\mathbf{r} - \mathbf{r}')} + e^{i\bar{\mathbf{k}}^{(-)} \cdot (\mathbf{r} - \mathbf{r}')} \right. \\ \left. - \frac{1}{\mathcal{A}} (\bar{r}_1 e^{i\bar{\mathbf{k}}^{(+)} \cdot \mathbf{r} - i\bar{\mathbf{k}}^{(-)} \cdot \mathbf{r}'} + \bar{r}_2 e^{i\bar{\mathbf{k}}^{(-)} \cdot \mathbf{r} - i\bar{\mathbf{k}}^{(+)} \cdot \mathbf{r}' - 2k_{\perp}d}) \right], \end{aligned} \quad (97)$$

where  $\mathcal{A} = 1 - \bar{r}_1(w) \bar{r}_2(w) \exp(-2k_{\perp}d)$ .

For the transverse Green's function, we set

$$\tilde{\mathbf{G}}_{\perp}^{(\text{cav})}(\mathbf{r}, \mathbf{r}', w) = \tilde{\mathbf{G}}_{\perp}^{(0)}(\mathbf{r} - \mathbf{r}', w) + \tilde{\mathbf{F}}_{\perp}^{(\text{cav})}(\mathbf{r}_{\perp} - \mathbf{r}'_{\perp}, z, z', w), \quad (98)$$

with

$$\tilde{\mathbf{F}}_{\perp}^{(\text{cav})} = \tilde{\mathbf{U}}^{(\text{cav})} + \tilde{\mathbf{V}}^{(\text{cav})}, \quad (99)$$

where the symbols have the obvious meaning, analogously to the previous section. The same arguments that led us to write equation (97) now give

$$\begin{aligned} \tilde{U}_{ij}^{(\text{cav})} = i \int \frac{d^2 \mathbf{k}_\perp}{2\pi k_z} \left\{ \left[ \left( \frac{1}{\mathcal{A}_s} - 1 \right) (e^{i\mathbf{k}^{(+)} \cdot (\mathbf{r}-\mathbf{r}')} + e^{i\mathbf{k}^{(-)} \cdot (\mathbf{r}-\mathbf{r}')} \right. \right. \\ \left. \left. + \frac{r_1^{(s)}}{\mathcal{A}_s} e^{i\mathbf{k}^{(+)} \cdot \mathbf{r} - i\mathbf{k}^{(-)} \cdot \mathbf{r}'} + \frac{r_2^{(s)}}{\mathcal{A}_s} e^{i\mathbf{k}^{(-)} \cdot \mathbf{r} - i\mathbf{k}^{(+)} \cdot \mathbf{r}' + 2ik_z d} \right] e_{\perp i} e_{\perp j} \right. \\ \left. + \frac{1}{k_0^2} \left[ \left( \frac{1}{\mathcal{A}_p} - 1 \right) (\xi_i^{(+)} \xi_j^{(+)} e^{i\mathbf{k}^{(+)} \cdot (\mathbf{r}-\mathbf{r}')} + \xi_i^{(-)} \xi_j^{(-)} e^{i\mathbf{k}^{(-)} \cdot (\mathbf{r}-\mathbf{r}')} \right. \right. \\ \left. \left. + \xi_i^{(+)} \xi_j^{(-)} \frac{r_1^{(p)}}{\mathcal{A}_p} e^{i\mathbf{k}^{(+)} \cdot \mathbf{r} - i\mathbf{k}^{(-)} \cdot \mathbf{r}'} + \xi_i^{(-)} \xi_j^{(+)} \frac{r_2^{(p)}}{\mathcal{A}_p} e^{i\mathbf{k}^{(-)} \cdot \mathbf{r} - i\mathbf{k}^{(+)} \cdot \mathbf{r}' + 2ik_z d} \right] \right\}, \end{aligned} \quad (100)$$

where  $r_i^{(\alpha)}$ ,  $\alpha = s, p$  are the Fresnel reflection coefficients of slab  $i$  for polarization  $\alpha$ , and  $\mathcal{A}_\alpha = 1 - r_1^{(\alpha)} r_2^{(\alpha)} \exp(2ik_z d)$ . For  $\tilde{\mathbf{V}}^{(\text{cav})}$  we obtain

$$\tilde{V}_{ij}^{(\text{cav})} = \frac{1}{k_0^2} \frac{\partial^2 \Psi^{(\text{cav})}}{\partial x_i \partial x_j'}, \quad (101)$$

where

$$\begin{aligned} \tilde{\Psi}^{(\text{cav})} = \int \frac{d^2 \mathbf{k}_\perp}{2\pi k_\perp} \left[ \left( \frac{1}{\mathcal{A}} - 1 \right) (e^{i\mathbf{k}^{(+)} \cdot (\mathbf{r}-\mathbf{r}')} + e^{i\mathbf{k}^{(-)} \cdot (\mathbf{r}-\mathbf{r}')} \right. \\ \left. - \frac{1}{\mathcal{A}} (\bar{r}_1 e^{i\mathbf{k}^{(+)} \cdot \mathbf{r} - i\mathbf{k}^{(-)} \cdot \mathbf{r}'} + \bar{r}_2 e^{i\mathbf{k}^{(-)} \cdot \mathbf{r} - i\mathbf{k}^{(+)} \cdot \mathbf{r}' - 2k_\perp d}) \right]. \end{aligned} \quad (102)$$

Again we find, as in one-slab case, that the expression of  $\tilde{\Psi}^{(\text{cav})}$  coincides with that of  $\tilde{F}^{(\text{cav})}$ . The same considerations used in the one-slab case can be now repeated for  $\tilde{F}^{(\text{cav})}$  and  $\tilde{\mathbf{F}}_\perp^{(\text{cav})}$  to show that both quantities are analytic in  $\mathcal{C}^+$  and have a finite limit for vanishing  $w$ .

### 5. Commutation relations for the em fields inside a cavity

In this section we compute the quantities  $A^{(\text{mat})}$ ,  $A_{\perp ij}^{(\text{mat})}$ ,  $B_{\perp ij}^{(\text{mat})}$  and  $C_{\perp ij}^{(\text{mat})}$  for the two slab setting considered in the previous section. The corresponding quantities shall be denoted by  $A^{(\text{cav})}$ ,  $A_{\perp ij}^{(\text{cav})}$ ,  $B_{\perp ij}^{(\text{cav})}$  and  $C_{\perp ij}^{(\text{cav})}$ , respectively. We shall see that they all vanish, as a consequence of the analyticity and fall-off properties at large frequencies of the reflection coefficients of all real materials. As seen in section 3, vanishing of these quantities entails that the e.m. field satisfies free-field commutation relations in the empty region between the slabs.

Consider first the quantity  $A^{(\text{cav})}(\mathbf{r}, \mathbf{r}')$ . From its definition (equation (51)) it follows that  $A^{(\text{cav})}$  can be expressed in terms of  $\tilde{F}^{(\text{cav})}$  as

$$A^{(\text{cav})}(\mathbf{r}, \mathbf{r}') = \lim_{\tau \rightarrow 0^+} \int_\Gamma \frac{dw}{2\pi} \tilde{F}^{(\text{cav})}(\mathbf{r}, \mathbf{r}', w) e^{-iw\tau}, \quad (103)$$

where  $\tilde{F}^{(\text{cav})}$  is given in equation (97). In appendix B it is shown that the reflection coefficient  $\bar{r}(w)$  of any real material vanishes like  $w^{-2}$  for large values of  $|w|$  and this implies, as can be seen by inspection of equation (97), that  $\tilde{F}^{(\text{cav})}$  approaches zero like  $w^{-3}$ . Therefore,  $\tilde{F}^{(\text{cav})}$  is absolutely integrable, and then in equation (103) we can take the  $\tau$ -limit inside the integral. After we do it we obtain

$$A^{(\text{cav})}(\mathbf{r}, \mathbf{r}') = \int_\Gamma \frac{dw}{2\pi} \tilde{F}^{(\text{cav})}(\mathbf{r}, \mathbf{r}', w). \quad (104)$$



The  $w^{-3}$  fall-off rate of  $F^{(\text{cav})}$  at infinity now permits us to close the integration contour in equation (104) in the upper complex  $w$ -plane  $\mathcal{C}^+$ , and then analyticity of  $\tilde{F}^{(\text{cav})}$  in  $\mathcal{C}^+$  implies at once that the integral is zero. Therefore, we conclude that

$$A^{(\text{cav})}(\mathbf{r}, \mathbf{r}') = 0. \quad (105)$$

We now turn to the quantity  $A_{\perp ij}^{(\text{cav})}(\mathbf{r}, \mathbf{r}')$ . In view of its definition (equation (52)) we have

$$A_{\perp ij}^{(\text{cav})}(\mathbf{r}, \mathbf{r}') = \lim_{\tau \rightarrow 0^+} \int_{\Gamma} \frac{dw}{2\pi} \tilde{F}_{\perp ij}^{(\text{cav})}(\mathbf{r}, \mathbf{r}', w) e^{-iw\tau}, \quad (106)$$

and then to prove that it vanishes, we need to consider the fall-off properties of  $\tilde{F}_{\perp ij}^{(\text{cav})}$ . According to equation (99), it is the sum of two terms:  $\tilde{F}_{\perp ij}^{(\text{cav})} = \tilde{U}_{\perp ij}^{(\text{cav})} + \tilde{V}_{\perp ij}^{(\text{cav})}$ . As to  $\tilde{V}_{\perp ij}^{(\text{cav})}$  we see, by inspection of equations (101) and (102), that the fall-off rate of  $\tilde{r}(w)$  implies that  $\tilde{V}_{\perp ij}^{(\text{cav})}$  falls-off like  $w^{-4}$ . Consider now  $\tilde{U}_{\perp ij}^{(\text{cav})}$ . We note first that, because of the  $k_z$  factors in the exponentials, all terms on the rhs of equation (100) decay exponentially fast as  $w$  goes to infinity in  $\mathcal{C}^+$  along any direction not parallel to the real axis. Along the real axis, since Fresnel reflection coefficients of all real materials decay like  $w^{-2}$  (see appendix B),  $\tilde{U}_{\perp ij}^{(\text{cav})}$  decays at least as fast as  $w^{-3}$  (in fact a more careful analysis carried out in appendix C shows that the rate of decay is actually  $w^{-4}$ ). Therefore,  $\tilde{F}_{\perp ij}^{(\text{cav})}$  decays in all directions in  $\mathcal{C}^+$  at least like  $w^{-3}$  and then, by following exactly the same reasoning used in the case of  $A^{(\text{cav})}(\mathbf{r}, \mathbf{r}')$ , we can prove that

$$A_{\perp ij}^{(\text{cav})}(\mathbf{r}, \mathbf{r}') = 0. \quad (107)$$

We remark that the above equation also holds when either  $\mathbf{r}$  or  $\mathbf{r}'$  or both belong to the slabs surfaces. Consider now the quantity  $B_{\perp ij}^{(\text{cav})}(\mathbf{r}, \mathbf{r}')$ . Recalling its definition (equation (53)), we have

$$B_{\perp ij}^{(\text{cav})}(\mathbf{r}, \mathbf{r}') = i \lim_{\tau \rightarrow 0^+} \int_{\Gamma} \frac{dw}{2\pi} w \tilde{F}_{\perp ij}^{(\text{cav})}(\mathbf{r}, \mathbf{r}', w) e^{-iw\tau}. \quad (108)$$

Thanks to the  $w^{-3}$  fall-off rate of  $\tilde{F}_{\perp ij}^{(\text{cav})}(w)$ , the extra power of  $w$  does not spoil convergence of the  $w$ -integral on the rhs of equation (108), and therefore the same arguments used to prove that  $A_{\perp ij}^{(\text{cav})}$  is zero can be used to obtain

$$B_{\perp ij}^{(\text{cav})}(\mathbf{r}, \mathbf{r}') = 0. \quad (109)$$

Finally, we consider the quantity  $C_{\perp ij}^{(\text{cav})}(\mathbf{r}, \mathbf{r}')$ . For this we have

$$C_{\perp ij}^{(\text{cav})}(\mathbf{r}, \mathbf{r}') = \lim_{\tau \rightarrow 0^+} \int_{\Gamma} \frac{dw}{2\pi} w^2 \tilde{F}_{\perp ij}^{(\text{cav})}(\mathbf{r}, \mathbf{r}', w) e^{-iw\tau}. \quad (110)$$

Proving that  $C_{\perp ij}^{(\text{cav})}$  vanishes requires much more labour, because of the *two* extra powers of  $w$  in the integrand on the rhs of equation (110). We relegate the proof in appendix C, where we show that the decay rate of  $\tilde{F}_{\perp ij}^{(\text{cav})}$  is actually  $w^{-4}$ , which is sufficiently fast to imply

$$C_{\perp ij}^{(\text{cav})}(\mathbf{r}, \mathbf{r}') = 0. \quad (111)$$

Having proved that the quantities  $A^{(\text{cav})}$ ,  $A_{\perp ij}^{(\text{cav})}$ ,  $B_{\perp ij}^{(\text{cav})}$  and  $C_{\perp ij}^{(\text{cav})}$  vanish, we then reach the important conclusion that in the empty space between two dielectric and/or conducting slabs, the e.m. fields satisfy free-field equal-time commutation relations, equation (12)–(17). This result is consistent with what was expected on the basis of the microscopic theory, for a system of non-relativistic charged particles interacting with the e.m. field, as we have seen in section 2. We remark that no singularities are encountered as  $\mathbf{r}$  and  $\mathbf{r}'$  approach the slabs surfaces, and therefore the canonical form of the free-space commutators also holds on the surfaces of the slabs. It is important to realize that these results are intimately tied to analyticity and fall-off properties of the reflection coefficients of real materials.

### 6. Commutation relations outside ideal conductors

In this section we investigate the commutation relations satisfied by the e.m. fields outside ideal conductors. Ideal conductors are characterized by the fact that they have *constant* reflection coefficients. Indeed, by taking the limit  $\epsilon \rightarrow \infty$  in equations (79), (84) and (85), we find that for an ideal conductor  $\bar{r}$  and  $r^{(p)}$  are 1, and  $r^{(s)}$  is  $-1$  at all frequencies. Obviously, constant reflection coefficients are analytic in  $\mathcal{C}^+$ , and therefore the main difference between ideal conductors and real ones is that the reflection coefficients of the former *do not* vanish in the limit of large frequencies. We shall see below that this feature entails that the e.m. field outside the conductors, and on their surfaces, fails to satisfy free-field canonical equal-time commutation relations.

In order to determine the commutation relations satisfied by the e.m. field we consider again equations (45)–(50) that remain valid also for ideal conductors. All that we have to do then is to evaluate the quantities on the rhs of these equations, using the values of the reflection coefficients pertaining to ideal conductors. We consider first the simpler case of a single conducting slab.

We start by evaluating the quantity  $F^{(\text{id wall})}$ , where the superscript (id wall) stands for a slab made of an ideal metal. From equation (77) we note that for  $\bar{r}(w) = 1$ ,  $\tilde{F}^{(\text{id wall})}$  becomes independent of the frequency and, upon taking the inverse time-Fourier transform, one easily finds that  $F^{(\text{id wall})}$  is proportional to  $\delta(t - t')$ . Then  $F^{(\text{id wall})}$  is zero for all  $t > t'$  and therefore from equation (35) we have

$$\langle [U(\mathbf{r}, t), U(\mathbf{r}', t')] \rangle = 0 \quad (\text{ideal conductors}). \tag{112}$$

Upon taking account also of equation (36) we see that outside an ideal conductor, all two-times commutators involving the scalar potential  $U$  have vanishing expectation values, and this implies

$$U(\mathbf{r}, t) \equiv 0 \quad (\text{ideal conductors}).$$

Therefore, outside an ideal conducting slab, the longitudinal electric field is zero. We evaluate now the quantity  $F_{\perp ij}^{(\text{id wall})}$ . Upon using the identity

$$-e_{\perp i} e_{\perp j} + \frac{\xi_i^{(+)} \xi_j^{(-)}}{k_0^2} = -\lambda_k \delta_{ik} \delta_{jk} + \frac{k_i^{(+)} k_j^{(-)}}{k_0^2}, \tag{113}$$

where  $\lambda_1 = \lambda_2 = -\lambda_3 = 1$ , one finds that, for  $\bar{r} = r^{(p)} = 1$  and  $r^{(s)} = -1$ , equations (81), (83) and (90) lead to

$$\begin{aligned} \tilde{F}_{\perp ij}^{(\text{id wall})} = & - \int \frac{d^2 \mathbf{k}_{\perp}}{2\pi} \left[ \frac{i}{k_z} \left( \lambda_k \delta_{ik} \delta_{jk} - \frac{k_i^{(+)} k_j^{(-)}}{k_0^2} \right) \right. \\ & \left. \times e^{ik_z(z+z')} + \frac{1}{k_{\perp}} \frac{\bar{k}_i^{(+)} \bar{k}_j^{(-)}}{k_0^2} e^{-k_{\perp}(z+z')} \right] e^{i\mathbf{k}_{\perp} \cdot (\mathbf{r}_{\perp} - \mathbf{r}'_{\perp})}. \end{aligned} \tag{114}$$

By a similar computation as the one described in appendix A, it is possible to verify that the rhs of the above equation can also be written in the following form:

$$\tilde{F}_{\perp ij}^{(\text{id wall})} = - \int \frac{d^3 \mathbf{k}}{2\pi^2} \frac{1}{k^2 - k_0^2} \left( \lambda_k \delta_{ik} \delta_{jk} - \frac{k_i k_j^{(r)}}{k^2} \right) e^{i[\mathbf{k}_{\perp} \cdot (\mathbf{r}_{\perp} - \mathbf{r}'_{\perp}) + k_3(z+z')]}, \tag{115}$$

where  $\mathbf{k}^{(r)} = \mathbf{k}_{\perp} - k_3 \hat{\mathbf{z}}$ . From this expression we see that in the case of an ideal wall  $\tilde{F}_{\perp ij}^{(\text{id wall})}$  decays for large frequencies only like  $w^{-2}$ , and not like  $w^{-4}$  as we found in the case of a

slab made of a real material. This fall-off rate is sufficient to prove, by the same steps used in the previous section, that equation (107) remains valid. Therefore, we find that also in the case of an ideal slab the equal-time commutators for the vector potential have the canonical form (equation (3)) at all points outside the slab, including its surface. The  $w^{-2}$  fall-off rate is not sufficient however to ensure validity of equation (109), and we show now that for an ideal conductor equation (109) indeed fails to be true. To see this we take the inverse Fourier transform of equation (115), as defined in equation (66). The frequency integral, for  $t > t'$ , can be easily evaluated by closing the contour  $\Gamma$  in the lower complex plane (which is possible now because the rhs of equation (115) is also analytic there), and by noting that the integrand has poles only at  $k_0 = \pm k$ . We get

$$F_{\perp ij}^{(\text{id wall})} = -c \int \frac{d^3 \mathbf{k}}{2\pi^2 k} \left( \lambda_k \delta_{ik} \delta_{jk} - \frac{k_i k_j^{(r)}}{k^2} \right) e^{i[\mathbf{k}_\perp \cdot (\mathbf{r}_\perp - \mathbf{r}'_\perp) + k_3(z+z')]} \sin[ck(t-t')]. \quad (116)$$

Then, from equation (49), we obtain

$$\langle [A_{\perp i}(\mathbf{r}, t), E_{\perp j}(\mathbf{r}', t)] \rangle = -4\pi i \hbar c \left[ \delta_{ij}^\perp(\mathbf{r} - \mathbf{r}') - \delta_{ij}^{(\text{id wall})}(\mathbf{r}_\perp - \mathbf{r}'_\perp, z + z') \right], \quad (117)$$

where we defined

$$\delta_{ij}^{(\text{id wall})}(\mathbf{r}_\perp - \mathbf{r}'_\perp, z + z') = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \left( \lambda_k \delta_{ik} \delta_{jk} - \frac{k_i k_j^{(r)}}{k^2} \right) e^{i[\mathbf{k}_\perp \cdot (\mathbf{r}_\perp - \mathbf{r}'_\perp) + k_3(z+z')]}. \quad (118)$$

We note that  $\delta_{ij}^{(\text{id wall})}$  is a smooth function for  $z + z' > 0$  approaching zero for large  $z$  and  $z'$ , but it is singular when both  $z$  and  $z'$  belong to the slab surface (i.e. for  $z = z' = 0$ ). In particular, for  $i = j = 1$ , equation (117) gives

$$\langle [A_{\perp 1}(\mathbf{r}, t), E_{\perp 1}(\mathbf{r}', t)] \rangle = -4\pi i \hbar c \left[ \delta_{11}^\perp(\mathbf{r} - \mathbf{r}') - \delta_{11}^\perp(\mathbf{r}_\perp - \mathbf{r}'_\perp, z + z') \right], \quad (119)$$

in agreement with the finding of [15]. By using equation (44), and equation (116), it is easy to verify that the canonical commutation relations for the components of the transverse electric field, equation (5), remain valid.

We turn now to the more elaborate case of two plane-parallel ideal slabs. We shall be brief here, the analysis being similar to the one-slab case. First we note that, similarly to  $\tilde{F}^{(\text{id wall})}$ , also the quantity  $\tilde{F}^{(\text{id cav})}$  becomes independent of the frequency when perfectly reflecting slabs are considered, as it is easily seen from equation (97). Therefore,  $F^{(\text{id cav})}$  is proportional to  $\delta(t-t')$ , and again we conclude that the scalar potential can be taken to be zero outside the slabs. We consider now the transverse Green's function. A somewhat lengthy but straightforward computation analogous to the one done for the one-slab case gives the following expression for the quantity  $F_{\perp ij}^{(\text{id cav})}$ :

$$\begin{aligned} F_{\perp ij}^{(\text{id cav})} = & -c \int \frac{d^3 \mathbf{k}}{2\pi^2 k} \frac{1}{\mathcal{A}^{(\text{id})}} \left[ \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) e^{ik_3(2d+z-z')} \right. \\ & + \left( \delta_{ij} - \frac{k_i^{(r)} k_j^{(r)}}{k^2} \right) e^{ik_3(2d-z+z')} + \left( \lambda_k \delta_{ik} \delta_{jk} - \frac{k_i k_j^{(r)}}{k^2} \right) \\ & \times e^{ik_3(z+z')} + \left. \left( \lambda_k \delta_{ik} \delta_{jk} - \frac{k_i^{(r)} k_j^{(r)}}{k^2} \right) e^{ik_3(2d-z-z')} \right] \\ & \times e^{i\mathbf{k}_\perp \cdot (\mathbf{r}_\perp - \mathbf{r}'_\perp)} \sin[ck(t-t')], \end{aligned} \quad (120)$$

where  $\mathcal{A}^{(\text{id})} = 1 - \exp(2ik_3 d)$ . By using this equation, and recalling equations (48) and (50), we easily see that the equal-time commutators for the vector potential on one hand and for the

transverse electric field on the other both vanish inside the cavity and on the slabs surfaces, in agreement with the free-field case, equation (3) and equation (5). On the other hand, from equation (49), we get

$$\langle [A_{\perp i}(\mathbf{r}, t), E_{\perp j}(\mathbf{r}', t)] \rangle = -4\pi i\hbar c \left[ \delta_{ij}^{\perp}(\mathbf{r} - \mathbf{r}') - \delta_{ij}^{(\text{id cav})}(\mathbf{r}_{\perp} - \mathbf{r}'_{\perp}, z, z') \right], \quad (121)$$

where

$$\begin{aligned} \delta_{ij}^{(\text{id cav})}(\mathbf{r}_{\perp} - \mathbf{r}'_{\perp}, z, z') &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{\mathcal{A}^{(\text{id})}} \left[ \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) \right. \\ &\quad \times e^{ik_3(2d+z-z')} + \left( \delta_{ij} - \frac{k_i^{(r)} k_j^{(r)}}{k^2} \right) e^{ik_3(2d-z+z')} + \left( \lambda_k \delta_{ik} \delta_{jk} - \frac{k_i k_j^{(r)}}{k^2} \right) e^{ik_3(z+z')} \\ &\quad \left. + \left( \lambda_k \delta_{ik} \delta_{jk} - \frac{k_i^{(r)} k_j}{k^2} \right) e^{ik_3(2d-z-z')} \right] e^{i\mathbf{k}_{\perp} \cdot (\mathbf{r}_{\perp} - \mathbf{r}'_{\perp})}. \end{aligned} \quad (122)$$

We note that the first and the second terms between the square brackets on the rhs of equation (122) represent smooth functions of  $z$  and  $z'$  at all points between the slabs, including their surfaces, while the third and fourth terms are singular, respectively, on the surface of slab 1 (i.e. for  $z = z' = 0$ ) and slab 2 (i.e. for  $z = z' = d$ ). Moreover, we observe that in the limit of large separations  $d$ , and for fixed  $z$  and  $z'$ , the phase factors involving  $d$  in the first, second and fourth terms between the square brackets on the rhs of equation (122), oscillate more and more rapidly, and so suppress the corresponding terms. In this limit  $\delta_{ij}^{(\text{id cav})}$  tends to  $\delta_{ij}^{(\text{id wall})}$ , and then equation (121) reproduces equation (117).

From the above analysis, we see that while all other commutators have the free-field form, the presence of the extra terms  $\delta_{ij}^{(\text{id wall})}$  and  $\delta_{ij}^{(\text{id cav})}$  on the rhs of equations (117) and (121), respectively, implies that perfect-mirror b.c. lead to equal-time commutation relations for the vector potential and the electric field of a different form from the free-field ones, equation (4). In the one-slab case, equations (117) and (118) show that the free-field form of the commutators is recovered only when the quantity  $\delta_{ij}^{(\text{id cav})}$  can be neglected, and this occurs at points  $z$  and  $z'$  that are far from the slab. In the cavity setting, equations (121) and (122) show that free-field commutation relations are recovered only provided that the quantities  $2d + z - z'$ ,  $2d - z + z'$ ,  $z + z'$  and  $2d - z - z'$  are simultaneously large. This is only possible for large cavities, and for points  $z$  and  $z'$  far from both conductors. This is in contrast with what was found in the previous section, where we proved that in the case of real materials free-fields equal-time canonical commutation relations retain their validity everywhere between the slabs, including on their surfaces.

A final comment is in order on gauge issues. As we noted in section 3, in the case of (nonmagnetic) dielectrics with sharp boundaries the transversality condition equation (2) defining the Coulomb gauge is valid at all points of space, including at points on the surfaces of the bodies. This is not true however for ideal conductors, as a result of the fact that the skin-depth of e.m. fields inside a perfect conductor is strictly zero. Since the vector potential vanishes inside a perfect conductor, it follows that differently from real dielectrics, which have a finite skin-depth, the normal component of the vector potential is discontinuous across the surface of a perfect conductor, and therefore its divergence has a Dirac-delta-type singularity at points lying on the surface of the conductor. Therefore, even if the vector potential satisfies the transversality condition, equation (2), at points outside the conductor, the Coulomb gauge condition is actually violated at points on its surface<sup>3</sup>. Since the commutation relations

<sup>3</sup> We thank an anonymous referee for drawing our attention to this point.

satisfied by the vector potential are *gauge-dependent*, one is then led to wonder if the obtained deviation from the canonical form of the equal-time commutation relations, satisfied by the vector potential in the case of perfect conductors, could be the result of the fact that the gauge we are considering does not really coincide with the Coulomb gauge of ordinary QED in this idealized case. It is easy to verify that this is not so, however, by considering the equal-time commutators between the magnetic and the electric fields, which are gauge-invariant. In the case of one slab, for example, by taking the curl with respect to  $\mathbf{r}$  of both sides of equation (117), one finds that the commutator of the magnetic field with the electric field deviates also from its canonical free-space form, as a result of the additional term, proportional to  $\delta_{ij}^{(\text{idwall})}$ , on the rhs of equation (117).

## 7. Concluding remarks

In this paper we have determined the commutation relations satisfied by the quantized e.m. field outside one or two plane-parallel dielectric and/or conducting slabs in vacuum, assuming that the slabs are made of isotropic and homogeneous, spatially non-dispersive materials, with arbitrary frequency-dependent dispersion and absorption. Using a general form of macroscopic quantum electrodynamics, we have found that at all points between the slabs, including on their surfaces, the e.m. field satisfies canonical commutation relations of the same form as in empty space, in full agreement with the microscopic theory. This result is a general consequence of analyticity and fall-off properties at large frequencies satisfied by the reflection coefficients of all real materials.

We have also shown that free-field equal-time commutation relations do not obtain outside one or two conducting slabs, if the latter are modelled as perfect mirrors, because of the extra terms that appear in the commutator of the vector potential with the electric field. Free-field commutators are only recovered at points that are sufficiently far from the mirror. In the one-slab setting, our findings coincide with those obtained by Milonni [15] in his investigation on the Casimir effect. Since no such deviation from free-field commutators is found in the case of real materials, we draw the conclusion that the modified form of the field commutation relations implied by perfect-mirror b.c. is an artefact of this idealized model. Even if the commutator of the vector potential with the electric field, being a gauge-dependent quantity, is not a physically observable quantity, failure of perfect mirror b.c. to reproduce the correct free-field form of the equal-time commutation relations near the surfaces of the conductors indicates that a certain amount of caution should be used when these idealized b.c. are used in investigations of proximity phenomena originating from the quantized e.m. field in the presence of conductors.

Before closing, we would like to comment on possible generalization of the results derived in this paper to other materials, including magnetic materials, and nonisotropic or spatially dispersive media. Consideration of isotropic magnetic materials offers no difficulties, because it just requires substituting in our formulae the well-known expression for Fresnel reflection coefficients, for a medium with magnetic permeability  $\mu$ . On the other hand, it is known today that the general formulae (equations (35)–(37)) expressing the expectation values of the field commutators in terms of their classical Green's functions, are valid for arbitrary media [25, 26], and therefore one can use them also in the case of anisotropic and/or spatially dispersive media provided only that one is able to determine the reflection coefficients for a slab made of these materials. Since reflection coefficients of all media are analytic functions of the complex frequency in the upper complex plane, and fall off to zero at large frequencies [24], it is therefore expected that (free-space) canonical commutation relations remain valid also for these more general materials.

### Appendix A. Weyl representation of the transverse Green's function in empty space

As it is well known, the time Fourier transform of Green's function for the transverse e.m. field in free space is given by the formula

$$\tilde{G}_{\perp ij}^{(0)} = \int \frac{d^3 \mathbf{k}}{2\pi^2} \frac{1}{k^2 - k_0^2} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) e^{i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')}. \quad (\text{A.1})$$

An expression analogous to Weyl's representation of the scalar Green's function, equation (68), can be obtained by performing the integral over  $k_3$  in equation (A.1). The integral can be done easily by suitably closing the  $k_3$  contour of integration in the complex  $k_3$  plane (for  $z \geq z'$  one closes the contour in the upper half-plane, for  $z \leq z'$  in the lower plane). The integral then receives contributions from poles in the integrand of equation (A.1), which arise from two sources. The first one is the factor involving the inverse of  $k^2 - k_0^2$ , which gives rise to poles at  $k_3 = \pm k_z$ , where  $k_z = \sqrt{k_0^2 - k_{\perp}^2}$  (the square root is defined such that  $\text{Im}(k_z) > 0$ ). Importantly, the second source of poles is the gauge-fixing term inside the round brackets, proportional to the inverse of  $k^2$ , which has poles at  $k_3 = \pm i k_{\perp}$ . We correspondingly split  $\tilde{\mathbf{G}}_{\perp}^{(0)}$  as the sum of two terms:

$$\tilde{\mathbf{G}}_{\perp}^{(0)} = \tilde{\mathbf{U}}^{(0)} + \tilde{\mathbf{V}}^{(0)}, \quad (\text{A.2})$$

where  $\tilde{\mathbf{U}}^{(0)}$  accounts for the former set of poles, and  $\tilde{\mathbf{V}}^{(0)}$  for the latter. We find

$$\tilde{U}_{ij}^{(0)} = \frac{i}{2\pi} \int \frac{d^2 \mathbf{k}_{\perp}}{k_z} \left( \delta_{ij} - \frac{k_i^{(\pm)} k_j^{(\pm)}}{k_0^2} \right) e^{i\mathbf{k}^{(\pm)} \cdot (\mathbf{r} - \mathbf{r}')}, \quad (\text{A.3})$$

where  $\mathbf{k}^{(\pm)} = \mathbf{k}_{\perp} \pm k_z \hat{\mathbf{z}}$  and

$$\tilde{V}_{ij}^{(0)} = \frac{1}{k_0^2} \int \frac{d^2 \mathbf{k}_{\perp}}{2\pi k_{\perp}} \bar{k}_i^{(\pm)} \bar{k}_j^{(\pm)} e^{i\bar{\mathbf{k}}^{(\pm)} \cdot (\mathbf{r} - \mathbf{r}')}, \quad (\text{A.4})$$

where  $\bar{\mathbf{k}}^{(\pm)} = \mathbf{k}_{\perp} \pm i k_{\perp} \hat{\mathbf{z}}$ , and in both equations (A.3) and (A.4) the upper (lower) sign is for  $z \geq z'$  ( $z \leq z'$ ). It is now convenient to further transform the expression of  $\tilde{U}_{ij}^{(0)}$  by considering the following decomposition of the identity  $\delta_{ij}$ :

$$\delta_{ij} = e_{\perp i} e_{\perp j} + \frac{1}{k_0^2} (\xi_i^{(\pm)} \xi_j^{(\pm)} + k_i^{(\pm)} k_j^{(\pm)}), \quad (\text{A.5})$$

where  $\mathbf{e}_{\perp} = \hat{\mathbf{z}} \times \hat{\mathbf{k}}_{\perp}$  and  $\xi^{\pm} = k_{\perp} \hat{\mathbf{z}} \mp k_z \hat{\mathbf{k}}_{\perp}$ . Upon replacing  $\delta_{ij}$ , in equation (A.3), by the rhs of equation (A.5), we obtain

$$\tilde{U}_{ij}^{(0)} = \frac{i}{2\pi} \int \frac{d^2 \mathbf{k}_{\perp}}{k_z} \left( e_{\perp i} e_{\perp j} + \frac{\xi_i^{(\pm)} \xi_j^{(\pm)}}{k_0^2} \right) e^{i\mathbf{k}^{(\pm)} \cdot (\mathbf{r} - \mathbf{r}')}. \quad (\text{A.6})$$

As for  $\tilde{V}_{ij}^{(0)}$ , we note that it represents a pure scalar contribution, for it can be written as

$$\tilde{V}_{ij}^{(0)} = \frac{1}{k_0^2} \frac{\partial^2 \tilde{\Psi}}{\partial x_i \partial x_j'}, \quad (\text{A.7})$$

where

$$\tilde{\Psi} = \int \frac{d^2 \mathbf{k}_{\perp}}{2\pi k_{\perp}} e^{i\bar{\mathbf{k}}^{(\pm)} \cdot (\mathbf{r} - \mathbf{r}')}. \quad (\text{A.8})$$

Upon combining equations (A.6) and (A.7), we obtain the following final expression for  $\tilde{G}_{\perp ij}^{(0)}$ :

$$\tilde{G}_{\perp ij}^{(0)} = \frac{i}{2\pi} \int \frac{d^2 \mathbf{k}_{\perp}}{k_z} \left( e_{\perp i} e_{\perp j} + \frac{\xi_i^{(\pm)} \xi_j^{(\pm)}}{k_0^2} \right) e^{i\mathbf{k}^{(\pm)} \cdot (\mathbf{r} - \mathbf{r}')} + \frac{1}{k_0^2} \frac{\partial^2 \tilde{\Psi}}{\partial x_i \partial x_j'}. \quad (\text{A.9})$$

### Appendix B. Properties of the reflection coefficients

In this appendix, we briefly review some general properties of the reflection coefficients of dielectrics and conductors that are important for the present paper. They are a direct consequence of the general properties of the electric permittivity  $\epsilon(\omega)$ . As it is well known [22], the permittivity  $\epsilon(\omega)$  of a causal medium is an analytic function of the complex frequency  $w$  in the upper complex plane  $C^+$ . Moreover, its imaginary part  $\epsilon''$  is never zero in  $C^+$ , except along the positive imaginary axis ( $w = i\xi, \xi > 0$ ), where  $\epsilon$  is positive and monotonically decreasing. For large (complex) frequencies  $w$ , the electric permittivities of all materials approach 1 and have an asymptotic expansion of the form [22]

$$\epsilon(w) = 1 - \frac{A}{w^2} + i\frac{B}{w^3} + \dots, \quad (\text{B.1})$$

where  $A$  and  $B$  are real positive constants characteristic of the material.

As a consequence of the above analyticity properties of  $\epsilon(w)$ , the reflection coefficients  $\bar{r}$ ,  $r^{(p)}$  and  $r^{(s)}$ , given by equations (79), (85) and (84), respectively, are analytic functions in  $C^+$ , and they are real and positive along the positive imaginary axis [24]. The asymptotic behaviour of  $\epsilon(w)$  implies that for large frequencies  $\bar{r}(w)$  and  $r^{(\alpha)}(w, k_{\perp})$ ,  $\alpha = s, p$  have asymptotic expansions of the form

$$\bar{r}(w) \simeq \frac{\bar{C}}{w^2} + i\frac{\bar{D}}{\omega^3} + \dots, \quad r^{(\alpha)}(w, k_{\perp}) \simeq \frac{C^{(\alpha)}}{w^2} + i\frac{D^{(\alpha)}}{w^3} + \dots, \quad (\text{B.2})$$

where  $\bar{C}$ ,  $\bar{D}$ ,  $C^{(\alpha)}$  and  $D^{(\alpha)}$  are real numbers characteristic of the material. We remark that  $C^{(\alpha)}$  and  $D^{(\alpha)}$  are independent of  $k_{\perp}$ .

It is useful to consider also the behaviour of the reflection coefficient in the limit of zero frequency. In the case of a dielectric,  $\epsilon(w)$  approaches a positive constant  $\epsilon_0 > 1$  at zero frequency, and therefore for the reflection coefficients  $\bar{r}(0)$ ,  $r^{(p)}(0)$  and  $r^{(s)}(0)$ , we find

$$\bar{r}(0) = r^{(p)}(0) = \frac{\epsilon_0 - 1}{\epsilon_0 + 1} < 1 \quad (\text{dielectrics}) \quad (\text{B.3})$$

$$r^{(s)}(0) = 0 \quad (\text{dielectrics}). \quad (\text{B.4})$$

Consider now conductors. At sufficiently low frequency, the permittivity of a conductor is of the form

$$\epsilon(w) = 4\pi i \frac{\sigma_0}{w} \quad (\text{conductors}), \quad (\text{B.5})$$

where  $\sigma_0$  is ohmic conductivity. Then for the reflection coefficients at zero frequency, we find

$$\bar{r}(0) = r^{(p)}(0) = 1 \quad (\text{conductors}), \quad (\text{B.6})$$

$$r^{(s)}(0) = 0 \quad (\text{conductors}). \quad (\text{B.7})$$

It is also useful to estimate the behaviour of the difference between  $r^{(p)}(w)$  and  $\bar{r}(w)$ , as  $w$  approaches zero. In the case of dielectrics, one finds

$$r^{(p)} - \bar{r} = \frac{\epsilon_0(\epsilon_0 - 1)}{(1 + \epsilon_0)^2} \frac{w^2}{c^2 k_{\perp}^2} + O(w^4), \quad (\text{B.8})$$

while, in the case of conductors, we find

$$r^{(p)} - \bar{r} = \frac{w^2}{c^2 k_{\perp}^2} + \frac{iw^3}{c^2 k_{\perp}^2 \sigma_0} \left( \frac{3}{4\pi} + \frac{\pi \sigma_0^2}{c^2 k_{\perp}^2} \right) + O(w^4). \quad (\text{B.9})$$

We see that in both cases, the difference  $r^{(p)} - \bar{r}$  approaches zero as  $w^2$ .

### Appendix C. Proof that $C_{\perp ij}^{(\text{cav})} = \mathbf{0}$

In this appendix we prove that the quantity  $C_{\perp ij}^{(\text{cav})}$  defined in equation (110) is zero.

Upon recalling equation (98), we first split  $C_{\perp ij}^{(\text{cav})}$  as

$$C_{\perp ij}^{(\text{cav})}(\mathbf{r}, \mathbf{r}') = D_{ij}(\mathbf{r}, \mathbf{r}') + E_{ij}(\mathbf{r}, \mathbf{r}'), \quad (\text{C.1})$$

where

$$D_{ij}(\mathbf{r}, \mathbf{r}') = \lim_{t \rightarrow t'^+} \frac{\partial^2 U_{ij}^{(\text{cav})}}{\partial t \partial t'}(\mathbf{r}, \mathbf{r}', t - t') \quad (\text{C.2})$$

and

$$E_{ij}(\mathbf{r}, \mathbf{r}') = \lim_{t \rightarrow t'^+} \frac{\partial^2 V_{ij}^{(\text{cav})}}{\partial t \partial t'}(\mathbf{r}, \mathbf{r}', t - t'). \quad (\text{C.3})$$

In terms of Fourier transforms, we can write the above quantities as

$$D_{ij} = \lim_{\tau \rightarrow 0^+} \int_{\Gamma} \frac{dw}{2\pi} w^2 \tilde{U}_{ij}^{(\text{cav})}(\mathbf{r}, \mathbf{r}', w) e^{-iw\tau}. \quad (\text{C.4})$$

$$E_{ij} = \lim_{\tau \rightarrow 0^+} \int_{\Gamma} \frac{dw}{2\pi} w^2 \tilde{V}_{ij}^{(\text{cav})}(\mathbf{r}, \mathbf{r}', w) e^{-iw\tau}. \quad (\text{C.5})$$

By the same arguments used earlier in the case  $A^{(\text{cav})}(\mathbf{r}, \mathbf{r}')$ , we see that  $D_{ij}$  and  $E_{ij}$  vanish, provided that  $\tilde{U}_{ij}^{(\text{cav})}$  and  $\tilde{V}_{ij}^{(\text{cav})}$  fall off like  $w^{-4}$  or faster. By inspection of equations (101) and (102), and recalling that  $\bar{r}(w)$  falls off like  $w^{-2}$ , we can easily see that this is the case for  $\tilde{V}_{\perp ij}^{(\text{cav})}$ . Therefore we have

$$E_{ij}(\mathbf{r}, \mathbf{r}') = 0. \quad (\text{C.6})$$

Consider now the quantity  $\tilde{U}_{ij}^{(\text{cav})}$  whose expression is provided by equation (100). In order to estimate the fall-off rate of the various terms on the rhs of equation (100), we need to recall that Fresnel reflection coefficients of all real materials decay like  $w^{-2}$  (see appendix B). Since the quantities  $(1/A_s - 1)$  and  $(1/A_p - 1)$  then decay like  $w^{-4}$ , we see that all terms involving these quantities on the rhs of equation (100) fall off at least like  $w^{-5}$  at large frequencies, and therefore they can be neglected. Consider now the remaining terms in the expression for  $\tilde{U}_{ij}^{(\text{cav})}(\mathbf{r}, \mathbf{r}', w)$ . Upon noticing that for large  $w$ ,  $\xi_i^{(\pm)} \xi_j^{(\pm)} = -\xi_i^{(\pm)} \xi_j^{(\mp)} = -\xi_i^{(\mp)} \xi_j^{(\pm)} = w^2 \hat{k}_{\perp i} \hat{k}_{\perp j} + O(w)$ , we get

$$\begin{aligned} \tilde{U}_{ij}^{(\text{cav})} = & i \int \frac{d^2 \mathbf{k}_{\perp}}{2\pi k_z} \left[ (r_1^{(s)} e^{i\mathbf{k}^{(+)} \cdot \mathbf{r} - i\mathbf{k}^{(-)} \cdot \mathbf{r}'} + r_2^{(s)} e^{i\mathbf{k}^{(-)} \cdot \mathbf{r} - i\mathbf{k}^{(+)} \cdot \mathbf{r}' + 2ik_z d}) e_{\perp i} e_{\perp j} \right. \\ & \left. - (r_1^{(p)} e^{i\mathbf{k}^{(+)} \cdot \mathbf{r} - i\mathbf{k}^{(-)} \cdot \mathbf{r}'} + r_2^{(p)} e^{i\mathbf{k}^{(-)} \cdot \mathbf{r} - i\mathbf{k}^{(+)} \cdot \mathbf{r}' + 2ik_z d}) k_{\perp i} k_{\perp j} \right] + O(w^{-4}). \end{aligned} \quad (\text{C.7})$$

Having reached this point, we take advantage of the fact that to order  $w^{-3}$  included, both Fresnel coefficients are independent of  $k_{\perp}$  (see equation (B.2)). By virtue of this, the rhs of equation (C.7) can be written as

$$\tilde{U}_{ij}^{(\text{cav})} = \frac{1}{w^2} \sum_{\alpha=s,p} \sum_{m=1,2} K_{m|ij}^{(\alpha)} I_m(w) + O(w^{-4}), \quad (\text{C.8})$$

where  $K_{m|ij}^{(\alpha)}$  are material-dependent constants, and

$$I_1(w) = i \int \frac{d^2 \mathbf{k}_{\perp}}{2\pi k_z} e^{i\mathbf{k}^{(+)} \cdot \mathbf{r} - i\mathbf{k}^{(-)} \cdot \mathbf{r}'}, \quad (\text{C.9})$$



while

$$I_2(w) = i \int \frac{d^2 \mathbf{k}_\perp}{2\pi k_z} e^{i\mathbf{k}^{(-)} \cdot \mathbf{r} - i\mathbf{k}^{(+)} \cdot \mathbf{r}' + 2ik_z d}. \quad (\text{C.10})$$

It is a simple matter to check that the integrals  $I_i$  can also be written as

$$I_1(w) = \int \frac{d^3 \mathbf{k}}{2\pi^2} \frac{1}{k^2 - k_0^2} e^{i[\mathbf{k}_\perp \cdot (\mathbf{r}_\perp - \mathbf{r}'_\perp) + k_3(z+z')]}, \quad (\text{C.11})$$

and

$$I_2(w) = \int \frac{d^3 \mathbf{k}}{2\pi^2} \frac{1}{k^2 - k_0^2} e^{i[\mathbf{k}_\perp \cdot (\mathbf{r}_\perp - \mathbf{r}'_\perp) + k_3(2d - z - z')]}. \quad (\text{C.12})$$

From this we see that both  $I_1$  and  $I_2$  fall off like  $w^{-2}$ , and therefore, in view of equation (C.8), we find that  $\tilde{U}_{ij}^{(\text{cav})}$  decays like  $w^{-4}$  or faster. Therefore,  $D_{ij}$  is zero, and then we obtain the desired result

$$C_{\perp ij}^{(\text{cav})}(\mathbf{r}, \mathbf{r}') = 0. \quad (\text{C.13})$$

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